Parahoric group schemes and nilpotent sections

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The setting

- let $\mathscr{A}$ be a complete discrete valuation ring, K its field of fractions, and $\mathfrak{f} = \mathscr{A}/\pi\mathscr{A}$ its residue field.
- note that we make no assumption about $\mathfrak{f}$ – it may be *imperfect*, for example.
- and let G be a connected and reductive linear alg group over K.
- write $\mathfrak{g} = \mathrm{Lie}(G)$ for the *Lie algebra* of G
- goal, put broadly: study the $G(\mathrm{K})$ orbits on $\mathcal{N}(\mathrm{K})$, where $\mathcal{N}$ is the variety of nilpotent elements in $\mathfrak{g}$.

Parahoric group schemes

- if G is *split* reductive over K – i.e. if G has a maximal torus T isomorphic to $\prod^d \mathbf{G}_m$ over K – there is a *reductive group scheme* $\mathscr{G}$ over $\mathscr{A}$ for which $G = \mathscr{G}_\mathrm{K}$.

  indeed, for any root datum, there corresponds a split reductive group scheme $\mathscr{G}_\mathbf{Z}$ over the integers; now take $\mathscr{G} = (\mathscr{G}_\mathbf{Z})_\mathscr{A}$.
- Bruhat-Tits defined a class of *smooth group schemes* $\mathscr{P}$ over $\mathscr{A}$ for which $\mathscr{P}_\mathrm{K} = G$ known as *parahoric group schemes.*

  When G is split, the split reductive group scheme $\mathscr{G}$ is one of these parahorics.
- Assume that G_L is split reductive for some unramified extension $\mathrm{K} \subset \mathrm{L}$.

  Then the unipotent radical $R_u\mathscr{P}_\mathfrak{f}$ is defined and split over $\mathfrak{f}$.

- **Theorem** (McNinch 2020) Assume that G_{L} is split reductive for some unramified extension $\mathrm{K} \subset \mathrm{L}$. Let $\mathscr{P}$ be a parahoric group scheme with $\mathscr{P}_{\mathrm{K}} = G$.

  Then there is a closed $\mathscr{A}$-subgroup scheme $\mathscr{M} \subset \mathscr{P}$ such that

  - $\mathscr{M}$ is a reductive group scheme over $\mathscr{A}$
  - $\mathscr{M}_{\mathfrak{f}}$ is a Levi factor of $\mathscr{P}_{\mathfrak{f}}$
  - $\mathscr{M}_{\mathrm{K}}$ is a reductive subgroup of G containing a maximal torus ("maximal rank reductive subgroup")

- In fact, $\mathscr{M}_{\mathrm{K}}$ is the centralizer of the image of a homomorphism $\mu \to G$ where $\mu = \lim \mu_n$

  in particular, the geometric conjugacy classes of the $\mathscr{M}_{\mathrm{K}}$ are described by sub-diagrams of the extended Dynkin diagram of G (description of *Borel-de Siebenthal*)

"Standard" groups

- when the characteristic is "nice enough", the *geometric* nilpotent orbits for G are "the same" as in char. 0.

- this holds for the class of *standard* reductive groups over a field (McNinch and Testerman 2016).

  - simple groups in *very good* characteristic are standard ("very good" means: p is good for G *and* the order of the co-center of G_{sc} is prime to p)

  - the class of standard groups is closed under separable isogeny

  - the class of standard groups is closed under taking connected centralizers of diagonalizable subgroups

  - $G \times T$ is standard $\iff$ G is standard, where T is a torus.

- I hope the terminology "standard groups" suggests a generalization of J.C. Jantzen's *standard hypotheses*

- over a field $\mathrm{GL}(V)$ is standard, while $\mathrm{SL}(V)$ is standard $\Leftrightarrow$ $\dim V \not\equiv 0 \pmod{p}$.

- If G is standard, $x \in G(\mathrm{K})$, and $X \in \mathrm{Lie}(G)(\mathrm{K})$ then $C_G(x)$ and $C_G(X)$ are *smooth* over K

Nilpotent orbits

- we'll say that a reductive group scheme $\mathscr{G}$ is *standard* if $\mathscr{G}_K$ and $\mathscr{G}_\mathfrak{k}$ are standard.
- For a parahoric group scheme $\mathscr{P}$, we say that $\mathscr{X} \in \mathrm{Lie}(\mathscr{P})$ is *a balanced nilpotent section* provided that
  - $\mathscr{X}_\mathrm{K}$ is nilpotent (so also $\mathscr{X}_\mathfrak{k}$ is nilpotent)
  - $C_\mathrm{K} = C_G(\mathscr{X}_\mathrm{K})$ and $C_\mathfrak{k} = C_{\mathscr{P}_\mathfrak{k}}(\mathscr{X}_\mathfrak{k})$ are smooth group schemes with $\dim C_\mathrm{K} = \dim C_\mathfrak{k}$.
- **Theorem** (McNinch 2008, 2021) Suppose that $\mathscr{G}$ is a standard reductive group scheme over $\mathscr{A}$ with $G = \mathscr{G}_\mathrm{K}$ and let $X \in \mathrm{Lie}(\mathscr{G}_\mathfrak{k})$ be a nilpotent element.

  Then there is a *balanced nilpotent section* $\mathscr{X} \in \mathrm{Lie}(\mathscr{G})$ with $\mathscr{X}_\mathfrak{k} = X$ *together with* an $\mathscr{A}$-homomorphism $\phi : \mathbf{G}_{m/\mathscr{A}} \to \mathscr{G}$ such that ϕ_K is a cocharacter associated with $\mathscr{X}_K$ and ϕ_k is a cocharacter associated with $\mathscr{X}_\mathfrak{k}$.
- target: obtain a similar result for general parahoric group schemes.

Existence of SL_2-homomorphisms

- let $\mathscr{G}$ be a standard reductive group scheme over $\mathscr{A}$, let $\mathscr{X} \in \mathrm{Lie}(\mathscr{G})$ be a balanced nilpotent section, and let $\phi : \mathbf{G}_{m/\mathscr{A}} \to \mathscr{G}$ be associated to $\mathscr{X}$ as in the conclusion of the previous Theorem.
- **Theorem** (McNinch 2021) If $\mathscr{X}_\mathfrak{k}^{[p]} = 0$ then there is a unique homomorphism $\Phi : \mathrm{SL}_{2,\mathscr{A}} \to \mathscr{G}$ such that
  - $\mathscr{X} = d\Phi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
  - $\phi(t) = \Phi \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$

Balanced sections for parahorics

- let $\mathscr{G}$ be standard reductive group scheme, and suppose that the characteristic p of $\mathfrak{k}$ satisfies $p > 2h - 2$ where h is the Coxeter number of the root datum of $G_{K_{\mathrm{alg}}}$.

  And let $\mathscr{P}$ be a parahoric group scheme for $G = \mathscr{G}_K$.
- **Theorem** (McNinch 2021) Suppose that $X_0 \in \mathrm{Lie}(\mathscr{P}_\mathfrak{k}/R)$ is nilpotent, where $R = R_u(\mathscr{P}_\mathfrak{k})$.

Then there is a balanced nilpotent section $\mathscr{X} \in \mathrm{Lie}(\mathscr{P})$ for which $\mathscr{X}_{\mathfrak{f}}$ has image X_0 in $\mathrm{Lie}(\mathscr{P}_{\mathfrak{f}}/R)$.

- construction:
  - first, use the Levi decomposition of $\mathscr{P}_{\mathfrak{f}}$ to find $X_1 \in \mathrm{Lie}(\mathcal{M}_{\mathfrak{f}}) \subset \mathrm{Lie}(\mathscr{P}_{\mathfrak{f}})$ with image X_0.
  - Since $\mathcal{M}$ is standard reductive group scheme, we may choose a balanced section $\mathscr{X} \in \mathrm{Lie}(\mathcal{M})$ with $\mathscr{X}_{\mathfrak{f}} = X_1$.
  - Characteristic assumptions imply $X_1^{[p]} = 0$. So we may find an SL_2-homomorphism $\Phi : \mathrm{SL}_{2/\mathscr{A}} \to \mathcal{M}$ for which

$$\mathscr{X} = d\Phi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  - the assumptions on p guarantee that the adjoint action of the image of Φ on each of the fibers of $\mathrm{Lie}(\mathscr{P})$ is *semisimple*, which permits the conclusion that $\mathscr{X}$ is balanced in $\mathrm{Lie}(\mathscr{P})$

Main consequence

- the assignment "$X_0 \to \mathscr{X}_{\mathrm{K}}$" is the same as the assignment given by DeBacker (DeBacker 2002).

  (the proof that this is so depends on some conjugacy results I'm suppressing here…)

- get a "Bala-Carter" like result, namely:
  - **Theorem** Suppose $p > 2h-2$. Let $X_1 \in \mathrm{Lie}(G)$ be nilpotent. Then there is a K-subgroup $M \subset G$ such that
    * M is an unramified reductive subgroup containing a maximal torus of G
    * $X_1 \in \mathrm{Lie}(M)$ is (geometrically) distinguished for the action of M.
  - "M is unramified" means that $M = \mathcal{M}_{\mathrm{K}}$ for a reductive group scheme $\mathcal{M}$. In particular, M has a maximal torus that splits over an unramified extension of K.

Example

- For $X_0 \in \text{Lie}(G)$ the ramification behavior of tori in $C_G^0(X)$ constrains the possible $\mathscr{P}$ for which there is $\mathscr{X} \in \text{Lie}(\mathscr{P})$ with $X_0 = \mathscr{X}_{\text{K}}$.
- Let $G = \text{Sp}(V)$ with $\dim V = 4m$ and suppose $p > 2$.
- and let L be an étale algebra of degree 2 over K.
- consider a torus $S = R^1_{\text{L/K}}\mathbf{G}_m$ in G such that geometrically S has precisely two weights on $V_{\text{K}_{\text{sep}}}$ and for which the weight spaces are maximal isotropic subspaces.
- there is a nilpotent element $X_0 \in \mathfrak{g}(\text{K})$ acting with partition $(2m, 2m)$ for which S is a Levi factor of $C_G(X)$.
- as above, there is $\mathscr{X}$ with $\mathscr{X}_{\text{K}} = X_0$ where:
  - L unramified $\Rightarrow \mathscr{X} \in \text{Lie}(\text{GL}_{2m/\mathscr{A}})$
  - L ramified $\Rightarrow \mathscr{X} \in \text{Lie}(\text{Sp}_{2m,\mathscr{A}} \times \text{Sp}_{2m,\mathscr{A}})$

Thanks for your attention!

Bibliography

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