Group cohomology and Levi decompositions of linear groups

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Linear algebraic groups

Let $\mathcal{F}$ be a field.

- **Basic example of a linear algebraic group:** The general linear group GL_n may be viewed as the open subvariety of the affine space $\mathbf{A}^{n^2} = \mathrm{Mat}_n$ of $n \times n$ matrices, defined by the non-vanishing of det. In particular, GL_n is an affine variety.
- An **algebraic group** G over $\mathcal{F}$ is a "group object in the category of $\mathcal{F}$-varieties".
- In more down-to-earth terms: the variety G should be a group, and multiplication $G \times G \to G$ and inversion $G \to G$ should be morphisms of varieties.
- Algebraic groups include for example such "non-linear" groups as elliptic curves over $\mathcal{F}$. But a **linear algebraic group** is an algebraic group which is an *affine variety.*

Linear algebraic groups

- **basic result**: G is a linear algebraic group iff it is a closed subgroup of GL_n for some $n \geq 1$.
- For any algebraic group G, one can consider the *group of rational points* $G(\mathcal{F})$, and more general the group of points $G(\Lambda)$ for any commutative $\mathcal{F}$-algebra Λ.
- From this point-of-view, an algebraic group G is a *functor* from the category of commutative $\mathcal{F}$-algebras to the category of groups.
- according to Hilbert's nullstellensatz, if $\mathcal{F}$ is algebraically closed, the $\mathcal{F}$-algebraic group G is determined by knowledge of the subgroup $G(\mathcal{F}) \subset \mathrm{GL}_n(\mathcal{F})$ (for suitable n).
- In general, the linear algebraic group is determined by its coordinate algebra $\mathcal{F}[G]$.
- For an extension field $\mathcal{F} \subset \mathcal{F}_1$, get a linear algebraic group $G_{\mathcal{F}_1}$ by **base change** - i.e. by using $\mathcal{F}[G] \otimes_{\mathcal{F}} \mathcal{F}_1$.

Linear algebraic groups: examples

Examples

- If A is a finite dimensional $\mathcal{F}$-algebra, the group of units $G = A^\times$ "is" a linear algebraic group via the rule $G(\Lambda) = (A \otimes_{\mathcal{F}} \Lambda)^\times$.
- If $A = \mathrm{End}_{\mathcal{F}}(V)$ for a finite dimensional $\mathcal{F}$-vector space V, we just recover $\mathrm{GL}(V) = \mathrm{GL}_n$ with $n = \dim V$.
- If W is a subspace of V, consider the algebra $B = \{X \in \mathrm{End}_{\mathcal{F}}(V) \mid XW \subset W\}$, and let $P = B^\times$ be the group of units.
- P is the stabilizer in $\mathrm{GL}(V)$ of the point $[W]$ for its action on the Grassmann variety $\mathrm{Gr}_d(V)$ where $d = \dim W$, and in fact the projective variety $\mathrm{Gr}_d(V)$ is isomorphic to $\mathrm{GL}(V)/P$.

The Lie algebra

- The Lie algebra of an algebraic group is the tangent space $\mathrm{Lie}(G) = T_1(G)$ at the identity; it is a linear space over $\mathcal{F}$.
- Consider the algebra $\mathcal{F}[\epsilon]$ of *dual numbers*, where $\epsilon^2 = 0$.
- The natural mapping $\mathcal{F}[\epsilon] \to \mathcal{F}$ with $\epsilon \mapsto 0$ determines a mapping $\pi : G(\mathcal{F}[\epsilon]) \to G(\mathcal{F})$, and one can identify $\mathrm{Lie}(G)$ as the kernel.
- (it remains to explain how to find the Lie bracket...)

Example:

Any element $g \in \mathrm{GL}_n(\mathcal{F}[\epsilon])$ in $\ker \pi$ has the form $I_n + \epsilon X$ for $X \in \mathrm{Mat}_n(\mathcal{F})$, so $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n = \mathrm{Mat}_n$.

Unipotent radicals – by example

Example: stabilizer of a subspace

Let again P be the stabilizer in $\mathrm{GL}(V)$ of the point $[W]$ for a sub-space $W \subset V$.

- Consider the subgroup of P defined by $R = \{X \in P \mid X_{|W} = 1_W \quad \text{and} \quad Xv \equiv v \pmod{W} \quad \forall v\}$
- As a group of matrices, we can describe R as follows:

$$R = \left\{ \begin{pmatrix} I_d & A \\ 0 & I_{n-d} \end{pmatrix} \mid A \in \mathrm{Mat}_{d,n-d} \right\}.$$

  every elt u of R has property: $u - 1_V$ is nilpotent. So R is "upper triangular with 1's on the diagonal." This is what is meant by a *unipotent subgroup*.
- R is a connected, normal subgroup of P of dimension $d(n-d)$, and $P/R \simeq \mathrm{GL}(W) \times \mathrm{GL}(V/W)$.

Reductive groups

A linear algebraic group G is **reductive** provided that $G_{\overline{\mathcal{F}}}$ has no normal connected unipotent subgroups of positive dimension, where $\overline{\mathcal{F}}$ is an alg closure.

Some reductive/non-reductive examples:

- The group $G = \mathrm{GL}(V)$ is reductive.
- non-reduc: group P of previous example has *unipotent radical* R and *reductive quotient* $P/R \simeq \mathrm{GL}(W) \times \mathrm{GL}(V/W)$.
- reductive: symplectic group $\mathrm{Sp}(V, \beta)$ where β is non-degenerate alternating form on V
- reductive: special orthogonal group $\mathrm{SO}(V, \beta)$ where β is non-degenerate symmetric form on V when $\mathcal{F}$ has char. different from 2.

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Levi factors

- The unipotent radical $R_{\overline{\mathcal{F}}}$ of $G_{\overline{\mathcal{F}}}$ is the maximal connected normal unipotent subgroup of G.
- If $\mathcal{F}$ is perfect, the following condition holds: **(R)** there is always an $\mathcal{F}$-subgroup $R \subset G$ for which $R_{\mathcal{F}}$ is the unipotent radical of $G_{\overline{\mathcal{F}}}$.
- When **(R)** holds; we say R is the unipotent radical of G.
- If **(R)** holds for G, an $\mathcal{F}$-subgroup $M \subset G$ is a *Levi factor* if the quotient mapping $\pi : G \to G/R$ induces an isomorphism $\pi_{|M} : M \to G/R$.
- Of course, $G \simeq R \rtimes M$ is then a semidirect product

Remark

We ignore in this talk the possibility that $R_{\overline{\mathcal{F}}}$ may fail to be defined over $\mathcal{F}$. For more on consequences of this (and more...!) see the text (Conrad, Gabber, and Prasad 2015).

Levi factors in char. 0

If char. of $\mathcal{F}$ is 0, G has a Levi factor. Indeed:

- First apply Levi's theorem to the finite dimensional Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ to find a semisimple Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{r}$ where $\mathfrak{r}$ is the radical of $\mathfrak{g}$.
- Now, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$, so that $\mathfrak{m}$ is an *algebraic Lie subalgebra* – see (Borel 1991).
- This condition means that there is a closed connected subgroup $M \subset G$ with $\mathrm{Lie}(M) = \mathfrak{m}$; evidently, M is semisimple.
- Choosing a maximal torus T_0 of M and a maximal torus T of G containing T_0, one finds that $\langle M, T \rangle = M.T$ is a reductive subgroup of G which is a complement to the unipotent radical R.
- **Moral:** "the Lie algebra is a pretty good approx. to G in char. 0"

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Groups with no Levi factor – via Witt vectors

When $\mathcal{F}$ has pos char, $\exists$ linear groups with no Levi factor.

- Let W = Witt vectors with residue field $W/pW = \mathcal{F}$.
- Can view $W_2 = W/p^2W$ as a "ring variety" over $\mathcal{F}$
- As a $\mathcal{F}$ variety, $W_2 \simeq \mathbf{A}^2$. Moreover, $W_2(\mathbf{F}_p) = \mathbf{Z}/p^2\mathbf{Z}$.
- In fact, viewing W_2 as a functor, can consider e.g. the functor $G(\Lambda) = \mathrm{GL}_n(W_2(\Lambda))$. This rule defines a linear algebraic group over $\mathcal{F}$ of dimension $2n^2$.
- If $n > 1$, have non-split exact sequence:

$$0 \to \mathrm{Lie}(\mathrm{GL}_n)^{[1]} \to G \to \mathrm{GL}_n \to 1$$

- unip rad is the vector group $R = \mathrm{Lie}(\mathrm{GL}_n)^{[1]}$; exponent indicates that action of G/R on R is "Frobenius twisted".

Cohomology

- Consider a linear representation V of G – given by homomorphism of alg groups $G \to \mathrm{GL}(V)$
- The *Hochschild cohomology* groups $H^\bullet(G, V)$ are the derived functor(s) of the fixed point functor $W \mapsto H^0(G, W) = W^G$ on the category of G-modules.
- can compute/describe using cocycles $Z^\bullet(G, V)$ which are *regular functions* $\prod^\bullet G \to V$.
- So for example the 2-cocycle $Z^2(G, V)$ are certain regular functions $G \times G \to V$ satisfying an appropriate condition.

Cohomology and group extensions

Consider an exact sequence

$$(\clubsuit) \quad 0 \to V \to E \xrightarrow{\pi} G \to 1$$

where E and G are linear algebraic groups and V is a linear representation of G viewed as a "vector group" – in particular, a unipotent algebraic group.

- Result of Rosenlicht implies – since V is split unipotent – that π has a section: there is a regular function $\sigma : G \to E$ with $\pi \circ \sigma = 1_G$.
- the assignment $(x, y) \mapsto \sigma(xy)^{-1}\sigma(x)\sigma(y)$ determines a regular 2 cocycle $\alpha_E : G \times G \to V$
- $\alpha_E \in Z^2(G, V)$ yields well-def class $[\alpha_E] \in H^2(G, V)$.
- The sequence $(\clubsuit)$ is split iff $[\alpha_E] = 0$.

More groups with no Levi factor

The preceding cohomology point-of-view leads to a construction of groups with no Levi factor:

- suppose M is a reduc gp, V an M-module and $\alpha \in Z^2(M, V)$.
- use α to define an extension group G_α

$$0 \to V \to G_\alpha \to M \to 1$$

Theorem

G_α has a Levi factor if and only if $0 = [\alpha] \in H^2(M, V)$.

- Thus to construct groups without Levi factors, you should seek out linear representations V of a reductive group M for which $H^2(M, V) \neq 0$.

Conjugacy of Levi factors

Suppose that G is a linear algebraic group, that V is a linear representation of G, and that

$$(\diamondsuit) \quad 0 \to V \to E \to G \to 1$$

is an exact sequence.

- $(\diamondsuit)$ determines a class $[\alpha] \in H^2(G, V)$ whose vanishing controls the splitting
- Assume $(\diamondsuit)$ is split and fix section $\sigma : G \to E$ which is homom of alg gps.
- $G_1 =$ image of $\sigma(G)$ is a complement to V in E.

Theorem

Suppose that $[\alpha] = 0$. If $H^1(G, V) = 0$, any two complements to V in E are conjugate by an element of $G(\mathcal{F})$.

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Linear actions on vector groups

If U is a vector group on which G acts, one says that the action is linear if there is a G-equivariant isomorphism of algebraic groups $U \simeq \mathrm{Lie}(U)$.

Assume that **(R)** holds for the linear algebraic group G with unipotent radical R. Suppose that R is *split unipotent*.

Theorem (McNinch 2014; D. Stewart if $\mathcal{F} = \overline{\mathcal{F}}$)

If G is connected, then R has a G-invariant filtration for which the successive quotients are vector groups with linear G-action.

Consequence:

Corollary

With G as above, assume $H^2(G, L) = 0$ for each composition factor L of $\mathrm{Lie}(R)$ as G-module. Then G has a Levi factor.

"Dimensional criteria"

Again assume G satisfies **(R)**. Let M be reduc quotient G/R of G, and assume M is *split reductive*. (Any reduc group is split if $\mathcal{F}$ is alg. closed).

Corollary (McNinch 2010)

Suppose that $\dim R \leq p$ *and that* $\operatorname{ch}\operatorname{Lie}(R) = \sum_{i=1}^{d} \operatorname{ch}(\nabla_i)$ *for some "standard M-modules"* $\nabla_i = H^0(\lambda_i) = H^0(M/B, \mathcal{L}_i)$. *Then* G *has a Levi factor.*

Indeed, use result (Jantzen 1997): any M-module V with $\dim V \leq p$ is semisimple. Thus the M-comp factors of $\operatorname{Lie}(R)$ are the ∇_i which are therefore simple, and then $H^2(M, \nabla_i) = 0$ for each i.

Another perspective

Consider group schemes $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{R}$ each smooth and of finite type over $\mathbf{Z}$.

- S'pose: $\mathcal{M}$ is split reductive $/\mathbf{Z}$ – i.e. $\mathcal{M}_{\mathcal{F}}$ is split reduc $\forall$ fields $\mathcal{F}$.
- and that $\mathcal{R}$ is unipotent – i.e. $\mathcal{R}_{\mathcal{F}}$ is unip $\forall$ $\mathcal{F}$.
- Finally suppose that there are homomorphisms $\mathcal{R} \to \mathcal{G} \to \mathcal{M}$ such that on base change to each field $\mathcal{F}$ we get an exact sequence:
$$1 \to \mathcal{R}_{\mathcal{F}} \to \mathcal{G}_{\mathcal{F}} \to \mathcal{M}_{\mathcal{F}} \to 1.$$

Proposition

There is a finite list of primes $S = \{p_1, \ldots, p_r\}$ with the following property: if the characteristic of the field $\mathcal{F}$ is not in S, then $\mathcal{G}_{\mathcal{F}}$ has a Levi factor.

"Trailer" for the subsequent lecture

- My interest in Levi factors initially arose while considering some linear algebraic groups that appear in the study of "reductive groups over local fields".
- The second talk will describe the groups I mean, and it will describe an existence theorem for Levi factors in that context.

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