Centralizers of nilpotent elements

George McNinch

Department of Mathematics
Tufts University

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The Bala-Carter Theorem

Let G be a connected and reductive group over a field k.

- A parabolic $P \subset G$ has a dense ("Richardson") orbit $\mathcal{O}$ on $\mathrm{Lie}(R_u P)$; $\mathcal{O}$ has a k-rational element X.
- for Richardson elts X, condition "X is *distinguished*" can be characterized via properties of P (P is a "distinguished parabolic").
- Let $X \in \mathrm{Lie}(G)$ be any nilpotent element.
- If X is not distiguished, choose $S \subset C_G(X)$ a max torus and let $L = C_G(S)$. Then $X \in Lie(L)$ is dist for L.

The Bala-Carter Theorem

- If G is *standard* there is a cocharacter $\phi : \mathbf{G}_m \to (L, L)$ such that $X \in \mathrm{Lie}(L)(\phi; 2)$.
- ϕ determines a (distinguished) parabolic $Q = Q(\phi) \subset L$.
- X is in the dense ("Richardson") Q-orbit on $\mathrm{Lie}(R_u Q)$.
- If k alg closed, assignment $X \mapsto (L, Q)$ gives a bij between nilpotent orbits and G-classes of such pairs "Levi subgroup L, distinguished parabolic $Q \subset L$".
- from the POV of combinatorics, this means that one can label nilpotent orbits using data related to root systems and their Dynkin diagrams.

Outline

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Setting of local fields

- Let $\mathrm{K} = \mathrm{Frac}(\mathcal{A})$ where $\mathcal{A}$ is a complete DVR and $\mathcal{A}/\pi\mathcal{A} = \mathrm{k}$.
- Want to study connected reductive G defined over K.
- And want to study nilpotent $G(\mathrm{K})$-orbits in $\mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathrm{K})$.
- When k is finite, these orbits play important role in harmonic analysis for locally compact group $G(\mathrm{K})$.
- Crucial question: if X and X' are *geometrically conjugate* - i.e. conjugate by an element of $G(\mathrm{L})$ for some extension field $\mathrm{K} \subset \mathrm{L}$, when are X and X' conjugate by $G(\mathrm{K})$?

Symplectic group example

Consider split symplectic group $G = \mathrm{Sp}(V, \sigma)$ where $\dim V = 4m$; suppose char $\mathrm{K} \neq 2$. ("Type C_{2m}")

- Let (W, β) and (U, γ) be vector spaces with non-degenerate forms, where β is symplectic and γ is symmetric. Suppose $\dim W = 2m$ and $\dim U = 2$.
- There is K-isometry $(V, \sigma) \simeq (W \otimes U, \beta \otimes \gamma)$.
- Let X_0 reg nilpotent in $\mathfrak{sp}(W)$ and check that $X_\gamma = X_0 \otimes 1_U \in \mathfrak{sp}(W \otimes U, \beta \otimes \gamma) \simeq \mathfrak{sp}(V, \sigma) = \mathrm{Lie}(G)$.

Symplectic group example

- partition of X_γ is $(2m, 2m)$; the Bala-Carter datum of X_γ is indep of γ.
- The reductive quotient M_γ of $C_G(X_\gamma)$ is orthog gp $O(U, \gamma)$
- if $\gamma = \gamma_{\mathsf{split}}$ is split, $M_\gamma^0 = \mathbf{G}_m$.
- if $\gamma = \gamma_{\mathrm{L}}$ is the norm form for quad ext $\mathrm{K} \subset \mathrm{L}$, then $M_\gamma^0 = R^1_{\mathrm{L}/\mathrm{K}} \mathbf{G}_m$.

Symplectic group - conclusion

- In particular, $X_{\gamma_{\text{split}}}$ and $X_{\gamma_{\mathrm{L}}}$ are not $G(\mathrm{K})$-conjugate for a quadratic field extension L of K.
- *Rough idea* (originating with Waldspurger and (DeBacker 2002)) If L is unramified over K, distinguish among between $X_{\gamma_{\text{split}}}$ and $X_{\gamma_{\mathrm{L}}}$ using data "over k". If L ramified over K, need to distinguish using *parahoric group schemes*.

Outline

Bala-Carter

Nilpotent orbits for groups over local fields

Parahoric group schemes

Split reductive group schemes over $\mathcal{A}$

- Suppose that G is split reductive over K.
- There is a split reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ with $\mathcal{G}_{\mathrm{K}} = G$ for which $\mathcal{G}_{\mathrm{k}}$ a reductive group over k with the same root datum as G.

- Theorem (McNinch 2017)
  Assume G and $\mathcal{G}_{\mathrm{k}}$ are "standard", and let $X_0 \in \mathrm{Lie}(\mathcal{G}_{\mathrm{k}})$ be nilpotent. Then there is a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})(\mathcal{A})$ such that
  (a) *$\mathcal{X}_{\mathrm{k}} = X_0$ and $\mathcal{X}_{\mathrm{K}}$ is nilpotent.*
  (b) *$\mathcal{X}_{\mathrm{k}}$ and $\mathcal{X}_{\mathrm{K}}$ have the same Bala-Carter datum.*
  (c) *the identity component of $C_{\mathcal{G}}(\mathcal{X})$ is smooth over $\mathcal{A}$.*

- (c) means in particular that $dim C_G(\mathcal{X}_{\mathrm{K}})$ and $\dim C_{\mathcal{G}_{\mathrm{k}}}(X_0)$ coincide.

Symplectic example - reductive parahoric

The symplectic group $\mathsf{Sp}(V,\sigma)$ is the generic fiber of the reductive $\mathcal{A}$-group scheme $\mathcal{G} = \mathsf{Sp}(\mathcal{L})$ where $\mathcal{L}$ is a $\mathcal{A}$-lattice in V for which $\sigma(\mathcal{L},\mathcal{L}) = \mathcal{A}$ and for which σ determines a non-degenerate form $\overline{\sigma}$ on $\overline{V} = \mathcal{L}/\pi\mathcal{L}$.

- $\mathcal{G}_{\mathrm{k}} = \mathsf{Sp}(\overline{V},\overline{\sigma})$.
- consider the nilpotent elements $X_{\mathsf{split}}, X_{\gamma_\ell;}$ in $\mathsf{Lie}(\mathcal{G}_{\mathrm{k}})$ where $\mathrm{k} \subset \ell$ is a separable quadratic ext,
- Let $\mathcal{X}_{\mathsf{split}}, \mathcal{X}_\ell \in \mathsf{Lie}(\mathcal{G})$ as in the preceding Theorem.
- If $\mathrm{K} \subset \mathrm{L}$ is the unramified ext realizing residue field ext $\mathrm{k} \subset \ell$, it is clear that $\mathcal{X}_{\ell,\mathrm{K}} = X_{\gamma_{\mathrm{L}}}$.
- On the other hand, if $\mathrm{K} \subset F$ is ramif quad ext, X_{γ_F} can't be the generic fiber of any section $\mathcal{X} \in \mathsf{Lie}(\mathcal{G})$ for which $C_{\mathcal{G}}(\mathcal{X})$ has smooth identity component.

Parahoric group schemes

If G is reductive over K, a parahoric group scheme attached to G is a smooth group scheme $\mathcal{P}$ over $\mathcal{A}$ with generic fiber $\mathcal{P}_{\mathrm{K}} = G$.

- In general $\mathcal{P}_{\mathrm{k}}$ is not reductive.
- (McNinch 2014) If G splits over a tamely ramified ext of K, $\mathcal{P}_{\mathrm{k}}$ has a Levi factor, at least *geometrically*.

- Theorem (McNinch 2018)

  If G splits over an unramified extension of K, there is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that $\mathcal{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$ and such that the reductive subgroup $\mathcal{M}_{\mathrm{K}}$ contains a maximal torus of G.

  - In fact, $\mathcal{M}_{\mathrm{K}}$ is - geometrically, at least - the centralizer of a homomorphism $\mu_N \to G$.

Nilpotent sections and parahoric group schemes

Let $\mathcal{P}$ be a parahoric group scheme attached to G, and suppose that G and $\mathcal{P}_{\mathrm{k}}$ are "standard" reductive groups.

- Spose G splits over unramif ext of K, and let $\mathcal{M} \subset \mathcal{P}$ as in the preceding Thm.
- Spose the residue char. $p > 2h - 2$ where h is the max of the Coxeter numbers of the components of Dynkin diagram of $G_{\overline{\mathrm{K}}}$.
- identify Lie($\mathcal{M}_{\mathrm{k}}$) with Lie algebra of reduc quot of $\mathcal{P}_{\mathrm{k}}$.
- let $X_0 \in$ Lie($\mathcal{M}_{\mathrm{k}}$) nilpotent.

- Theorem (McNinch 2017)
  There is a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M}) \subset \mathrm{Lie}(\mathcal{P})$ *such that*
  (a) $\mathcal{X}_{\mathrm{k}} = X_0$ *and* $\mathcal{X}_{\mathrm{K}}$ *is nilpotent*
  (b) *the identity component of* $C_{\mathcal{P}}(\mathcal{X})$ *is smooth over* $\mathcal{A}$.

Symplectic parahoric example

- There is a parahoric group scheme $\mathcal{P}$ attached to Sp_{4m} for which $\mathcal{P}_{\mathrm{k}}$ has reductive quotient $\mathsf{Sp}_{2m/\mathrm{k}} \times \mathsf{Sp}_{2m/\mathrm{k}}$. ("$C_m \times C_m \subset \widetilde{C_{2m}}$")
- A reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in the preceding theorem has generic fiber $M = \mathcal{M}_{\mathrm{K}} \simeq \mathsf{Sp}_{2m/\mathrm{K}} \times \mathsf{Sp}_{2m/\mathrm{K}}$.
- Let $X_0 = (X_{\mathrm{reg}}, X_{\mathrm{reg}}) \in \mathfrak{sp}_{2m,\mathrm{k}} \times \mathfrak{sp}_{2m,\mathrm{k}}$.
- Choose $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ as in previous theorem. Then $\mathcal{X}_{\mathrm{K}}$ is $X_{\gamma_{\mathrm{L}}}$ for ramified quadratic extension $\mathrm{K} \subset \mathrm{L}$.

Symplectic parahoric example, redux

- There is a parahoric group scheme $\mathcal{P}$ attached to Sp_{4m} for which $\mathcal{P}_{\mathrm{k}}$ has reductive quotient GL_{2m}. ("$A_{2m-1} \subset C_{2m}$")
- If $\mathcal{M} \subset \mathcal{P}$ is as before, then $\mathcal{M}_{\mathrm{K}} = \mathsf{GL}_{2m}$.
- If X_0 is regular in $\mathsf{Lie}(\mathcal{M}_{\mathrm{k}})$, then $\mathcal{X}_{\mathrm{K}}$ is X_{split}.

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