Centralizers of nilpotent elements

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Special session at Bowdoin - September 2016
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Introduction

This talk will describe some applications of “comparison results” for centralizers of nilpotent elements in the Lie algebra of a linear algebraic group.

Part of the results described appear in the joint paper mcninch16:MR3477055 in Proc. AMS with Donna Testerman (EPFL).

The second part describes an improved version of a result from mcninch08:MR2423832; it will appear in mcninch16: nilpotent-orbits-over-local-field.
Standard reductive groups

We want to define a notion of *standard* reductive groups over a field $\mathcal{F}$:

- Semisimple groups in “very good” characteristic are standard, and tori are standard.
- If $G$ is standard and $H$ is *separably isogenous* to $G$, then $H$ is also standard.
- If $G_1$ and $G_2$ are standard, so is $G_1 \times G_2$.
- If $D \subset G$ is a diagonalizable subgroup scheme and if $G$ is standard, then also $C_G^0(D)$ is standard.
- In particular: $\text{GL}_n$ is standard for all $n \geq 1$.
- If $G$ is standard and if $L$ is a Levi factor of a parabolic of $G$, then $L$ is standard.
- Not standard: symplectic or orthogonal groups in char. 2.
Standard reductive groups: properties

Suppose that $G$ is a standard reductive group over the field $\mathcal{F}$.

**Theorem**

(a) *The center $Z$ of $G$ (as a group scheme) is smooth over $\mathcal{F}$. 
(b) *The centralizers $C_G(X)$ and $C_G(x)$ are smooth over $\mathcal{F}$ for every $X \in \text{Lie}(G)$ and every $x \in G(\mathcal{F})$. 
(c) *There is a $G$-invariant nondegenerate bilinear form on $\text{Lie}(G)$. 
(d) *There is a $G$-equivariant isomorphism – a Springer isomorphism – $\varphi : \mathcal{U} \rightarrow \mathcal{N}$ where $\mathcal{U} \subset G$ is the unipotent variety and $\mathcal{N} \subset G$ is the nilpotent variety.

**Theorem (mcninch09:MR2497582)**

For $X \in \text{Lie}(G)$ and $x \in G(\mathcal{F})$, $Z(C_G(X))$ and $Z(C_G(x))$ are smooth over $\mathcal{F}$. 


Nilpotent elements for a standard reductive group over a field

- Let $G$ a “standard” reductive alg gp over the field $\mathcal{F}$.
- Let $X \in \text{Lie}(G)$ nilpotent. A cocharacter $\phi : \mathbb{G}_m \rightarrow G$ is associated to $X$ if $X \in \text{Lie}(G)(\phi; 2)$ and if $\phi$ takes values in $(M, M)$ where $M = C_G(S)$ for a maximal torus $S \subset C_G(X)$.

**Theorem**

(a) There are cocharacters associated to $X$ (“defined over $\mathcal{F}$”).

(b) Any two cocharacters associated to $X$ are conjugate by an element of $U(\mathcal{F})$ where $U = R_u C_G(X)$.

(c) Each cocharacter $\phi$ associated to $X$ determines the same parabolic subgroup $P = P(\phi)$. In fact,

$$\text{Lie}(P) = \sum_{i \geq 0} \text{Lie}(G)(\phi; i).$$
Nilpotent elements: associated cocharacters

Let $X$ nilpotent and let $\phi$ be a cocharacter associated to $X$.

- If $\mathcal{F}$ has characteristic 0, let $(Y, H, X)$ be an $\mathfrak{sl}_2$-triple containing $X$. Then up to conjugacy by $U(\mathcal{F})$, $\text{Lie}(G)(\phi; i)$ is the $i$-eigenspace of $\text{ad}(H)$.

- For general $\mathcal{F}$, we have the following result:

**Theorem (mcninch05:MR2142248)**

If $X^{[p]} = 0$ there is a unique $\mathcal{F}$-homomorphism $\psi : \text{SL}_2,\mathcal{F} \to G$ such that $d\psi(E) = X$ and $\psi_S = \phi$, where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and where $S \simeq G_m$ is the diagonal torus of $\text{SL}_2$. 
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Even nilpotent elements

$G$ is a standard reductive group over $\mathcal{F}$ and $X \in \text{Lie}(G)$ nilpotent.

- Let $\phi$ be a cocharacter associated to $X$.
- $X$ is even if $\text{Lie}(G)(\phi; i) \neq 0 \implies i \in 2\mathbb{Z}$.
- If $X$ is even, then $\dim C_G(X) = \dim M$ where $M = C_G(\phi)$ is a Levi factor of $P = P(\phi)$. 
Main result

Theorem (mcninch16:MR3477055)

If $X$ is even, $\dim Z(C_G(X)) \geq \dim Z(M)$. [Where $Z(-)$ means “the center of -”].

- In fact, Lawther-Testerman already proved that equality holds (for $G$ semisimple). Their methods were “case-by-case”.
- The argument I’ll describe here is more direct.
- Reason for interest: let the unipotent $u$ correspond to $X$ via a Springer isomorphism. In char. $p > 0$, one has in general no well-behaved exponential map, but one might still hope to embed $u$ in a “nice” abelian connected subgroup. $Z(C_G(X))^0 = Z(C_G(u))^0$ is a starting point.
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Reductions

One knows that

$$\text{Lie}(Z(C_G(X))) = \mathfrak{z}(\text{Lie}(C_G(X))^\text{Ad}(B) = \mathfrak{z}(c_g(X)) \cap g^\text{Ad}(B)$$

where $B = C_{C_G(X)}(\phi)$.

In particular, to prove the main result, it is enough to argue that $\dim \mathfrak{z}(c_g(X)) \cap g^\text{Ad}(B) \geq \dim \mathfrak{z}(\text{Lie}(M))$.

(This reduction requires to know: the center of the standard reductive group $M$ is smooth!)

Let $A = k[T] \subset K = k(T)$. For simplicity of exposition, we note here if the char. of $k$ is 0, a proof of the Theorem can be given by studying the center of the centralizer of $X + TH$ in $\text{Lie}(G) \otimes_k A$. We now sketch some of this argument.
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Modules over a Dedekind domain

- Let $A$ be a Dedekind domain – e.g. a principal ideal domain.
- For a maximal ideal $m \subset A$ and an $A$-module $N$, write $k(m) = A/m$, and $N(m) = N/mN = N \otimes_A k(m)$,
- let $K$ be the field of fractions of $A$ and write $N_K = N \otimes_A K$.
- Let $M$ be a fin. gen $A$-module. Then $M = M_0 \oplus M_{tor}$ where $M_{tor}$ is torsion and $M_0$ is projective.
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Homomorphisms (notation)

- Let $\phi : M \rightarrow N$ be an $A$-module homom where $M$ and $N$ are f.g. projective $A$-modules.
- Let $P = \ker \phi$ and $Q = \coker \phi$.
- Write $Q = Q_0 \oplus Q_{\text{tor}}$ as before.
- $M/P$ is torsion free and thus projective, so for any max’l ideal $m$, we may view $P(m)$ as a subspace of $M(m)$.
- Write $\phi(m) : M(m) \rightarrow N(m)$ for $\phi \otimes 1_{k(m)}$.  

Fibers of a kernel

Recall $\phi : M \to N$, $P = \ker \phi$, and $Q = \coker \phi$.

**Theorem**

(a) $P(m) \subset \ker \phi(m)$, with equality $\iff Q_{\text{tor}} \otimes k(m) = 0$.

(b) $P(m) = \ker \phi(m)$ for all but finitely many $m$.

- Pf of (a) uses the following fact: for a finitely generated $A$-module $M$

  $$(\clubsuit) \quad \operatorname{Tor}_A^1(M, k(m)) \cong M_{\text{tor}} \otimes k(m)$$

- For (b), one just notes that $Q_{\text{tor}}$ has *finite length*.

- If one knows that $\dim_{k(m)} \ker \phi(m)$ is equal to a constant $d$ for all $m$ in some infinite set $\Gamma$ of prime ideals, then $d = \dim_K \ker \phi(K)$. 

Fibers of the center of an $A$-Lie algebra

- Let $L$ be a Lie algebra over $A$ which is f.g. projective as $A$-module.
- Let $Z = \{X \in L \mid [X, L] = 0\}$ be the center of $L$.

**Theorem**

(a) $L/Z$ is torsion free.
(b) $\dim_{k(m)} Z(m)$ is constant.
(c) For each maximal $m \subset A$, $Z(m) \subset z(L(m))$, and equality holds for all but finitely many $m$.

- Here $z(L(m))$ means the center of the $k(m)$-Lie algebra $L(m)$.
- The result essentially follows from the result for kernels.
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Center example

Let $A = k[T]$ for alg. closed $k$, and identify maximal ideals of $A$ with elements in $k$.

Let $L = Ae + Af$, with $e$ and $f$ an $A$-basis where $[e, f] = T \cdot f$.

Now $Z(L) = 0$, and $z(L(t)) = 0$ for $t \neq 0$.

But $L(0)$ is abelian, i.e. $z(L(0)) = L(0)$. 
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Center of the centralizer

Return to the setting of even nilpotent $X \in \mathfrak{g}$.

- Write $D = c_{\mathfrak{g}A}(X + T \cdot H)$.
- Write $Z$ for the center of the $A$-Lie algebra $D$.
- And write $H = \mathfrak{g}^B \otimes A \subset L$.
- Ultimately, must argue that

$$(Z \cap H)(1) \subset \mathfrak{z}(c_{\mathfrak{g}}(X)) \cap \mathfrak{g}^B$$

while for almost all $t \neq 1$,

$$(Z \cap H)(t) = Z(t) = c_{\mathfrak{g}}(X + tH).$$

- This implies the “main result”.

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Let \( A \) be a complete discrete valuation ring with field of fractions \( K \) and residue field \( k \).

Let \( G \) be a reductive \( A \)-group scheme with connected fibers \( G_K \) and \( G_k \).

The fibers \( G_K \) and \( G_k \) are reductive linear algebraic groups. The group scheme \( G \) is affine, smooth, and of finite type over \( A \).

Since \( G \) is smooth over \( A \), \( \text{Lie}(G) \) is a projective (hence free) \( A \)-module of finite rank.

If \( X \in \text{Lie}(G) \) and if \( X_K \) is nilpotent in \( \text{Lie}(G_K) \), then also \( X_k \) is nilpotent, and we say that \( X \) is a *nilpotent section*. 
Balanced sections

- Consider a $G$-module $\mathcal{L}$ which is free of finite rank as $A$-module.
- Given $X \in \mathcal{L}$, one can form the scheme theoretic stabilizer $C = \text{Stab}_G(X)$. Then $C$ is a group scheme over $A$, and we have

$$C_K = \text{Stab}_{G_K}(X_K) \quad \text{and} \quad C_k = \text{Stab}_{G_k}(X_k).$$

- We say that $X$ is balanced for the action of $G$ if $C_K$ is smooth over $K$, if $C_k$ is smooth over $k$, and if $\text{dim } C_K = \text{dim } C_k$. 
Recognizing balanced sections

Proposition (mcninch16:nilpotent-orbits-over-local-field)

Let \( X \in \mathcal{L} \). Write \( \mathfrak{g} = \text{Lie}(\mathcal{G}) \), and assume the following:

(a) the \( \mathcal{G}_K \) orbit of \( X_K \) is smooth – i.e.
\[
\dim \text{Stab}_{\mathcal{G}_K}(X_K) = \dim_K \mathfrak{c}_{\mathfrak{g}_K}(X_K),
\]
and
(b) \( \dim_K \mathfrak{c}_{\mathfrak{g}_K}(X_K) = \dim_k \mathfrak{c}_{\mathfrak{g}_k}(X_k) \).

Then \( X \) is balanced for the action of \( \mathcal{G} \).

The main points are: (i) \( \dim C_K \geq \dim C_k \) by Chevalley’s upper semicontinuity theorem, and (ii) smoothness on the generic fiber implies that \( \dim C_K \) coincides with the dimension of the stabilizer of \( x_K \) in \( \mathfrak{g}_K \).
Now suppose that the fibers $G_K$ and $G_k$ are standard reductive groups, that $\mathcal{L} = \text{Lie}(G)$ is the adjoint $G$-module, and let $X \in \text{Lie}(G)$.

Then the centralizer in $G_K$ of $X_K$ and the centralizer in $G_k$ of $X_k$ are automatically smooth, so $X$ is balanced if and only if the Lie algebraic centralizers on the fibers have the same dimension.
Existence and conjugacy of balanced nilpotent sections.

**Theorem (mcninch16:nilpotent-orbits-over-local-field)**

Let $X_0 \in \text{Lie}(G_k)$ nilpotent.

(a) There is a balanced, nilpotent section $X \in \text{Lie}(G)$ s.t. that $X_k \in \text{Lie}(G_k)$ coincides with $X_0$.

(b) There is an $A$-homom $\phi : G_m \to G$ s.t. $X \in \text{Lie}(G)(\phi; 2)$, $\phi_k$ is a cochar assoc with $X_k$ and $\phi_K$ is a cochar assoc with $X_K$.

(c) Let $X, X' \in \text{Lie}(G)$ be balanced nilpotent sections with $X_k = X'_k = X_0$. Then there is an element $g \in G(A)$ such that $X' = \text{Ad}(g)X$.

- The “Bala-Carter data” of $X_K$ and $X_k$ are “the same”.
- Using results of mcninch16:reductive-subgroup-schemes, the result is extended in mcninch16:nilpotent-orbits-over-local-field to so-called parahoric group schemes (under some further assumptions).
Theorem

Let $X \in \text{Lie}(G)$ be a balanced nilpotent section and let $\phi : G_m \to G$ be an $A$-homomorphism such that $\phi_F$ is a cocharacter associated to $X_F$ for $F \in \{k, K\}$. If $(X_k)^{[p]} = 0$, there is a unique $A$-homomorphism

$$\Phi : \text{SL}_2/A \to \mathcal{M}$$

such that $d\Phi(E) = X$, and $\Phi|_S = \phi$, where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Lie}(\text{SL}_2,A)$$

and $S \simeq G_m,A$ is the diag torus of $\text{SL}_2$. 
Smoothness

Theorem (Brian Conrad)

Let $\mathcal{H}$ be a group scheme of finite type over $\mathcal{A}$ for which the fibers $\mathcal{H}_K$ and $\mathcal{H}_k$ are each smooth of the same dimension. Then there is a locally closed subgroup scheme $\mathcal{M} \subset \mathcal{H}$ such that:

(a) $\mathcal{M}$ is smooth, affine, and of finite type over $\mathcal{A}$,
(b) $\mathcal{M}_K = (\mathcal{H}_K)^0$ and $\mathcal{M}_k = (\mathcal{H}_k)^0$.

Corollary

If $X \in \text{Lie}(\mathcal{G})$ is balanced section, there is a locally closed subgroup scheme $\mathcal{M} \subset C = C_{\mathcal{G}}(X)$ such that:

- $\mathcal{M}$ is smooth, affine and of finite type over $\mathcal{A}$, and
- $\mathcal{M}_K = C_{\mathcal{G}_K}^0(X_K)$ and $\mathcal{M}_k = C_{\mathcal{G}_k}^0(X_k)$
Smoothness, continued

- In mcninch08:MR2423832, it was claimed that $C = C_g(X)$ is smooth when $X$ is balanced, but the argument is incorrect (it fails to justify why $C$ is *flat* over $\mathcal{A}$).
- Results on the previous slide essentially fix the problem for the identity component $C^0$.
- However, with knowing the smoothness of the “full centralizer group scheme” $C$, the given arguments for mcninch08:MR2423832 are incorrect. That theorem concerns a comparison of the component groups $C_K/C_K^0$ and $C_k/C_k^0$. I don’t know whether the conclusion of the Theorem is correct.
The reductive quotient of a nilpotent centralizer

Theorem (Theorem A of \textit{mcninch08:MR2423832})

Let $X \in \text{Lie}(\mathcal{G})$ a balanced nilpotent section. The geom root datum of the reduc quotient of the conn centralizer $C_{\mathcal{G}_K}^0(X_K)$ is the same as the geom root datum of the reduc quotient of $C_{\mathcal{G}_k}^0(X_k)$.

- This proof can be found in \textit{mcninch16:nilpotent-orbits-over-local-field}.
- In fact, let $\phi : \mathbb{G}_m \to \mathcal{G}$ be an $\mathcal{A}$-homom s.t. $\phi_K$ is a cochar assoc to $X_K$ and $\phi_k$ is a cochar assoc to $X_k$.
- And let $\mathcal{M} \subset C$ be the smooth locally closed subgp scheme of the Corollary above.
- Then the centralizer $L = C_{\mathcal{M}}(\phi)$ is a reductive subgroup scheme of $\mathcal{M}$ for which $L_K$ is a Levi factor of $C_{\mathcal{G}_K}^0(X_K)$ and $L_k$ is a Levi factor of $C_{\mathcal{G}_k}^0(X_k)$.
- Now use: $\mathcal{M}$ splits over some unramified extension of $\mathcal{A}$. 
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