SUB-PRINCIPAL HOMOMORPHISMS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let G be a reductive group over an algebraically closed field of characteristic p, and let $u \in G$ be a unipotent element of order p. Suppose that p is a good prime for G. We show in this paper that there is a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ whose image contains u. This result was first obtained by D. Testerman (J. Algebra, 1995) using case considerations for each type of simple group (and using, in some cases, computer calculations with explicit representatives for the unipotent orbits).

The proof we give is free of case considerations (except in its dependence on the Bala-Carter theorem). Our construction of ϕ generalizes the construction of a principal homomorphism made by J.-P. Serre in (Invent. Math. 1996); in particular, ϕ is obtained by reduction modulo $\mathfrak{p}$ from a homomorphism of group schemes over a valuation ring $\mathscr{A}$ in a number field. This permits us to show moreover that the weight spaces of a maximal torus of $\phi(\mathrm{SL}_{2/k})$ on $\mathrm{Lie}(G)$ are "the same as in characteristic 0"; the existence of a ϕ with this property was previously obtained, again using case considerations, by Lawther and Testerman (Memoirs AMS, 1999) and has been applied in some recent work of G. Seitz (Invent. Math. 2000).

1. INTRODUCTION

Let $G = G_{/k}$ be a connected reductive algebraic group over an algebraically closed field k of characteristic $p > 0$. It is the main goal of this note to give another proof of the following theorem:

Theorem 1. *(Testerman* [Tes95]*) Suppose that p is a good prime for G. If $u \in G$ is unipotent and has order p, then there is a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ with u in its image.*

One might regard Theorem 1 as a group analogue of the Jacobson-Morozov theorem for Lie algebras. If one considers instead any field E of characteristic 0, a reductive group $G_{/E}$ over E, and $u \in G_{/E}$ an E-rational unipotent element, one may write $u = \exp(X)$ for a nilpotent $X \in \mathrm{Lie}(G_{/E})$; from the Jacobson-Morozov theorem for $\mathrm{Lie}(G_{/E})$ one deduces a homomorphism $\mathrm{SL}_{2/E} \to G_{/E}$ over E with u in its image.

Testerman's original proof of Theorem 1 used case considerations for each type of simple group (and used, in some cases, computer calculations with explicit unipotent class representatives known from the work of Mizuno). Our proof for the most part avoids case considerations (except that it depends on Pommerening's proof of the Bala-Carter theorem in good characteristic). We will exploit a weak Jacobson-Morozov-type result for an integral form of the Lie algebra of G. We obtain ϕ as a

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suitable G-conjugate of the reduction mod $\mathfrak{p}$ of a homomorphism of group schemes $\phi_{/\mathscr{A}} : \mathrm{SL}_{2/\mathscr{A}} \to G_{/\mathscr{A}}$, where $\mathscr{A}$ is a valuation ring in a number field.

When u is regular unipotent and has order p, the theorem yields a *principal homomorphism*; see [Ser96, §2]. The argument we give specializes in the regular case to the proof of *loc. cit.* Proposition 2.

The fact that we obtain a homomorphism of group schemes over $\mathscr{A}$ permits us to prove a more precise version of Theorem 1, which was first obtained by Lawther and Testerman.

In good characteristic, one can associate to a nilpotent element $X \in \mathfrak{g}$ a cocharacter ν, which is well defined up to conjugation by $C_G^o(X)$ (the connected centralizer); see Proposition 6. Moreover, we may find a G-equivariant homeomorphism ε from the nilpotent variety to the unipotent variety; see Proposition 29. We say that ε is a Springer homeomorphism.

We say that a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ is *sub-principal* [1] if the restriction of ϕ to a maximal torus of $\mathrm{SL}_{2/k}$ is a cocharacter associated to some non-0 nilpotent element X in the image of $d\phi$, and if $\varepsilon(X)$ is conjugate to a unipotent element in the image of ϕ.

Note that by Lemma 28 and the Bala-Carter Theorem (Proposition 3), if ε' is another Springer homeomorphism, then $\varepsilon(X)$ is conjugate to $\varepsilon'(X)$. Thus the notion of a sub-principal homomorphism is independent of this choice.

We may now state the more precise form of Theorem 1:

Theorem 2. *(Lawther and Testerman* [LT99, Theorem 4.2]*) With the assumptions of Theorem 1, there is a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ such that u is in the image of ϕ, the restriction of ϕ to a maximal torus of* $\mathrm{SL}_{2/k}$ *is a co-character associated to some nilpotent $0 \neq X \in$ Image$(d\phi)$, and $\varepsilon(X)$ is conjugate to u. Thus ϕ is a* sub-principal *homomorphism.*

In the language used by Lawther and Testerman [LT99], the theorem yields an A_1 subgroup of G whose labeled Dynkin diagram is the same as the labeled diagram of u. Indeed, the labeled diagram of the A_1 subgroup is obtained by choosing a maximal torus T_0 of $\mathrm{SL}_{2/k}$ and maximal torus T of G containing $\phi(T_0)$. The homomorphism $\mu = \phi_{|T_0} : T_0 \simeq \mathbf{G}_m \to G$ is then a cocharacter. For a suitable choice of Borel subgroup B containing T (equivalently: a suitable choice of positive roots) the values $\langle \alpha, \mu \rangle$ at the simple roots in $X^*(T)$ are non-negative and constitute the labels on the Dynkin diagram. One checks that these labels are independent of the choices made; see [Hu95, §7.6]. Similarly, the labels on the diagram of u are the non-negative integers $\langle \alpha, \nu \rangle$ where ν is a co-character associated to u (where again T and B are suitably chosen).

Now let E be an algebraically closed field of characteristic 0, and let $G_{/E}$ be a reductive group over E with the same root datum as G. There is a bijection between unipotent classes in G and unipotent classes in $G_{/E}$ which preserves labeled diagrams. The Dynkin-Kostant classification of nilpotent orbits in characteristic 0 implies: If $\phi_{/E} : \mathrm{SL}_{2/E} \to G_{/E}$ is any homomorphism with the unipotent element

[1] To explain this terminology, note that the main result of [LT99] shows (under some conditions on p which are slightly more restrictive than "p is good") that there is a unique conjugacy class of principal homomorphisms ϕ with $d\phi \neq 0$, and that for each unipotent class there is a unique conjugacy class of sub-principal homomorphisms (in the sense defined in this paper). Thus the classes of sub-principal homomorphisms are in some sense analogous to the class of principal homomorphisms (they are precisely the classes which "come from characteristic 0").

v in its image, the labeled diagram of $\phi_{/E}$ coincides with that of v. Thus Theorem 2 yields a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ whose image contains u and for which the weights of a maximal torus of $\mathrm{SL}_{2/k}$ on $\mathrm{Lie}(G)$ are "the same as in characteristic 0." (We refer the reader to the extensive tables in [LT99] to see that for some u there are homomorphisms $\phi : \mathrm{SL}_{2/k} \to G$ whose image contains u, but whose labeled diagram differs from that of u).

We mention that Theorem 2 was used by Seitz in [Sei00]. In *loc. cit.*, Seitz introduced the notion of a "good A_1". In the language above, he calls a homomorphism $\phi : \mathrm{SL}_2(k) \to G$ *good* if each weight $\lambda \in \mathbf{Z}$ of the representation $(\mathrm{Ad} \circ \phi, \mathfrak{g})$ for a maximal torus of $\mathrm{SL}_2(k)$ satisfies $\lambda \le 2p - 2$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Seitz proves that for each unipotent u of order p, there is a good homomorphism ϕ with u in its image; his proof of the existence of such a ϕ depends in a crucial way on Theorem 2 (when u is distinguished, the existence of a good ϕ is an immediate consequence of Theorem 2 combined with the "order formula" of Testerman which one will find in [Tes95] or [M02, Theorem 1.1]).

We also obtain a refinement of Theorem 2 for finite fields: if G is defined over a finite field $\mathbf{F}_q$ of good characteristic and u is $\mathbf{F}_q$-rational, we show in §5.3 that ϕ may be defined over $\mathbf{F}_q$ as well. In the course of our proof, we establish the following result which may be of independent interest. Let G be defined over an arbitrary field F (of good characteristic) and let u be an F-rational unipotent element. Suppose that either the orbit of u is separable, or that F is perfect. Then the canonical parabolic subgroup attached to u is defined over F. (The same statement holds for F-rational nilpotent elements).

Finally, we present two appendices. In the first, we give a proof that in good characteristic, there is always a G-equivariant *homeomorphism* between the nilpotent variety $\mathcal{N}$ and the unipotent variety $\mathcal{U}$ of a reductive group. Of course, in "very good" characteristic, there is an isomorphism of varieties due to Springer; our argument handles also groups such as $\mathrm{PGL}_{p/k}$ in characteristic p. This simplifies some of the steps in our proof of Theorem 2. In the second, we show that the G-equivariant isomorphism $\mathcal{U} \simeq \mathcal{N}$ defined by Bardsley and Richardson [BR85] respects the p-th power operations.

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2. Generalities on reductive groups

Let $G_{/\mathbf{Z}}$ be a split reductive group scheme over $\mathbf{Z}$. If $G_{/\mathbf{Z}}$ is moreover semisimple, one may regard $G_{/\mathbf{Z}}$ as a "Chevalley group scheme" as in [Bor70]. Let $\mathfrak{g}_{\mathbf{Z}}$ be the Lie algebra. For any commutative ring Λ, we put $G_{/\Lambda} = G_{/\mathbf{Z}} \times_{\mathbf{Z}} \Lambda$, and $\mathfrak{g}_\Lambda = \mathfrak{g}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \Lambda$.

Let $(X, Y, R, R^\vee)$ be the root datum of $G_{/\mathbf{Z}}$ with respect to a fixed maximal torus $T_{/\mathbf{Z}}$. Fix a $\mathbf{Z}$-basis $\gamma_1, \dots, \gamma_r$ for Y, the co-character group of $T_{/\mathbf{Z}}$. Let $H_{\gamma_i} = d\gamma_i(1) \in \mathfrak{h}_{\mathbf{Z}}$, the Lie algebra of $T_{/\mathbf{Z}}$. The algebra $\mathfrak{g}_{\mathbf{Z}}$ has a Chevalley basis

$$\{E_\alpha \mid \alpha \in R\} \cup \{H_{\gamma_1}, \dots, H_{\gamma_r}\}.$$

We have $\mathfrak{h}_{\mathbf{Z}} = \bigoplus_i \mathbf{Z} H_{\gamma_i}$, and $\mathfrak{b}_{\mathbf{Z}} = \mathfrak{h}_{\mathbf{Z}} \oplus \bigoplus_{\alpha \in R^+} \mathbf{Z} E_\alpha$ is a Borel subalgebra of $\mathfrak{g}_{\mathbf{Z}}$.

2.1. Good primes.

Recall the notion of a good prime for the root system R (or for the root datum $(X, Y, R, R^\vee)$, it is the same).

For the indecomposable root systems, a prime is bad (= not good) only in the following situations: 2 is bad unless R is of type A, 3 is bad if R is of type E, F or G, and 5 is bad if $R = E_8$.

For general R, p is good for R provided that it is good for each indecomposable component of R.

2.2. **Parabolic subalgebras.** If S denotes the simple roots in R^+, any subset $I \subset S$ determines a subroot system R_I in a well-known way, and hence a *standard parabolic subalgebra*

$$\mathfrak{p}(I)_{\mathbf{Z}} = \mathfrak{b}_{\mathbf{Z}} \oplus \bigoplus_{\alpha \in R_I^+} \mathbf{Z} E_{-\alpha}.$$

Consider the function $f : \mathbf{Z}R \to \mathbf{Z}$ which satisfies

$$f(\alpha) = \begin{cases} 2 & \alpha \in S \setminus I \\ 0 & \alpha \in I \end{cases} \tag{1}$$

We may regard f as a co-character of the adjoint group, so that $\mathfrak{g}_{\mathbf{Z}}$ becomes a module via f for $\mathbf{G}_{m/\mathbf{Z}}$; as such, it is the direct sum of its weight spaces. Thus, we have $\mathfrak{g}_{\mathbf{Z}} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(i)$, where

$$\mathfrak{g}_{\mathbf{Z}}(0) = \mathfrak{h}_{\mathbf{Z}} \oplus \bigoplus_{f(\alpha)=0} \mathbf{Z} E_\alpha, \quad \text{and} \quad \mathfrak{g}_{\mathbf{Z}}(i) = \bigoplus_{f(\alpha)=i} \mathbf{Z} E_\alpha \quad \text{for } i \neq 0.$$

We obtain the original parabolic algebra as $\mathfrak{p}(I)_{\mathbf{Z}} = \bigoplus_{i \geq 0} \mathfrak{g}_{\mathbf{Z}}(i)$. The opposite parabolic subalgebra is $\mathfrak{p}(I)^-_{\mathbf{Z}} = \bigoplus_{i \leq 0} \mathfrak{g}_{\mathbf{Z}}(i)$. We put $\mathfrak{u}(I)_{\mathbf{Z}} = \bigoplus_{i > 0} \mathfrak{g}_{\mathbf{Z}}(i)$ and $\mathfrak{u}(I)^-_{\mathbf{Z}} = \bigoplus_{i < 0} \mathfrak{g}_{\mathbf{Z}}(i)$.

There are "group scheme versions" of each of these constructions: i.e. there are parabolic subgroup schemes $P(I)_{/\mathbf{Z}}$ and $P(I)^-_{/\mathbf{Z}}$ with respective subgroup schemes $U(I)_{/\mathbf{Z}}$ and $U(I)^-_{/\mathbf{Z}}$.

2.3. **Distinguished nilpotents and parabolics.** Let k be an algebraically closed field with characteristic $p \geq 0$; in this section and the next we write $G = G_{/k}$ and $\mathfrak{g} = \mathfrak{g}_k$. We suppose that p is good for G. A nilpotent element $X \in \mathfrak{g}$, respectively a unipotent element $u \in G$, is said to be *distinguished* if the connected center of G is a maximal torus of $C_G(X)$, respectively $C_G(u)$. A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called *distinguished* if

$$\dim \mathfrak{p}/\mathfrak{u} = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] + \dim \operatorname{Lie}(Z),$$

where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$ and Z is the center of G.

Let $\mathfrak{p}_{\mathbf{Z}} = \mathfrak{p}(I)_{\mathbf{Z}}$ be a standard parabolic subalgebra of $\mathfrak{g}_{\mathbf{Z}}$ as in §2.2, and let $\mathfrak{u}_{\mathbf{Z}} = \mathfrak{u}(I)_{\mathbf{Z}}$. Then $\mathfrak{u} = \mathfrak{u}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$ is the nilradical of $\mathfrak{p} = \mathfrak{p}_Z \otimes_{\mathbf{Z}} k$. Let $\mathfrak{g}(i) = \mathfrak{g}_{\mathbf{Z}}(i) \otimes_{\mathbf{Z}} k$ for $i \in \mathbf{Z}$. Since p is good, $\dim_k \mathfrak{g}(2) = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$; see [M02, Prop. 4.3] or the proof of [Car93, Prop. 5.8.1].

Thus the condition that $\mathfrak{p}_k$ be distinguished is independent of k so long as the characteristic of k is good; we say that $\mathfrak{p}_{\mathbf{Z}}$ is a distinguished standard parabolic subalgebra if $\mathfrak{p}_k$ is distinguished.

When $\mathfrak{p}_{\mathbf{Z}}$ is distinguished, it follows from [J, Lemma 5.2] that the map $f : \mathbf{Z}R \to \mathbf{Z}$ of (1) extends uniquely to a linear function $X^*(T_{/\mathbf{Z}}) \to \mathbf{Z}$ and hence determines a cocharacter τ of $T_{/\mathbf{Z}}$ satisfying:

$$\langle \alpha, \tau \rangle = f(\alpha). \tag{2}$$

Note that the argument in *loc. cit.* applies to semisimple G, which we may reduce to by considering the derived group of G.

2.4. **Richardson orbits and the Bala-Carter Theorem.** Let k, G, $\mathfrak{g}$ as in the previous section; especially, recall that p is good. Suppose now that $\mathfrak{p} \subset \mathfrak{g}$ is any parabolic subalgebra, with nilradical $\mathfrak{u}$. There is a unique parabolic subgroup $P \leq G$ with $\mathrm{Lie}(P) = \mathfrak{p}$. Moreover, $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of P. A theorem of R. Richardson [Hu95, Theorem 5.3] says that there is a nilpotent G-orbit $\mathcal{O} \subset \mathfrak{g}$ with the property that $\mathcal{O} \cap \mathfrak{u}$ is an open P-orbit. Similarly, there is a unipotent class $\mathcal{C}$ in G with the property that $\mathcal{C} \cap U$ is an open P-orbit. By a Richardson element, we mean an orbit representative for $\mathcal{C}$ or $\mathcal{O}$ lying in U respectively $\mathfrak{u}$. The orbits $\mathcal{O}$ and $\mathcal{C}$ are known as the Richardson orbits associated with $\mathfrak{p}$ (or with P).

By a Levi subgroup of G, we mean a Levi factor of a parabolic subgroup.

Proposition 3. *(Bala-Carter, Pommerening* [Pom77, Pom80]*)*

(1) *Consider the collection of all pairs $(L, \mathcal{O})$ consisting of a Levi subgroup of G and a distinguished nilpotent orbit in* $\mathrm{Lie}(L)$. *Then the map which associates to $(L, \mathcal{O})$ the G-orbit* $\mathrm{Ad}(G)\mathcal{O}$ *defines a bijection between the set of G-orbits of pairs $(L, \mathcal{O})$ and the nilpotent G-orbits in $\mathfrak{g}$.*
(2) *Associate to each distinguished parabolic subalgebra its nilpotent Richardson orbit. Then this map defines a bijection between the conjugacy classes of distinguished parabolic subalgebras and the distinguished nilpotent orbits in $\mathfrak{g}$.*

Note that (1) holds with no assumption on p, but that (2) requires p to be good.

A cocharacter $\nu : \mathbf{G}_m \to G$ is said to be associated to a nilpotent element $X \in \mathfrak{g}$ provided that $\mathrm{Ad}(\nu(t))X = t^2 X$ for all $t \in \mathbf{G}_m$, and that ν takes values in the derived group of some Levi subgroup L for which $X \in \mathrm{Lie}(L)$ is distinguished.

Remark 4. Let L be a Levi subgroup of a parabolic in G, and let X be a nilpotent element in $\mathrm{Lie}(L)$. If $\iota : L \to G$ is the inclusion map, then a cocharacter τ of L is associated to X (with respect to L) if and only if $\iota \circ \tau$ is associated to X (with respect to G). If $\varepsilon : \mathcal{N} \to \mathcal{U}$ is a G-equivariant homeomorphism, then Lemma 28 (in the appendix) shows that ε restricts to a suitable L-equivariant homeomorphism. It follows that $\phi : \mathrm{SL}_{2/k} \to L$ is a sub-principal homomorphism (for L) if and only if $\iota \circ \phi$ is a sub-principal homomorphism (for G).

Remark 5. Let $\pi : G \to G'$ be a central isogeny of reductive groups. According to Lemma 27, $d\pi$ restricts to a G-equivariant homeomorphism between the respective nilpotent varieties. If L is a Levi subgroup of G then $\pi(L) = L'$ is a Levi subgroup of G', and it follows that X is distinguished in $\mathrm{Lie}(L)$ if and only if $d\pi(X)$ is distinguished in $\mathrm{Lie}(L')$. So if X is nilpotent in $\mathrm{Lie}(G)$, and if ϕ is a co-character associated to X, then $\pi \circ \phi$ is a co-character associated to $d\pi(X)$.

We record the following:

Proposition 6. *Suppose the characteristic is good for G. There is a cocharacter associated to any nilpotent element X. Moreover, any two cocharacters associated to X are conjugate by an element of $C_G^o(X)$.*

Proof. [J, Lemma 5.3] □

Proposition 7. *Suppose the characteristic is good for G. Let X be nilpotent, and let ν be a cocharacter associated to X. Consider the parabolic subalgebra $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, and let P be the corresponding parabolic subgroup of G.*

(1) $C_G(X) < P$.
(2) *P and $\mathfrak{p}$ depend only on X.*

Proof. [J, Prop. 5.9] □

The subalgebra $\mathfrak{p}$ is known as the *canonical* (or *Jacobson-Morozov*) parabolic determined by X.

2.5. $\mathfrak{sl}_2$ **triples.** Let Λ be an integral domain. The data $(0,0,0) \neq (X,Y,H) \in \mathfrak{g}_\Lambda \times \mathfrak{g}_\Lambda \times \mathfrak{g}_\Lambda$ is called an $\mathfrak{sl}_2$ triple (over Λ) if the Λ-linear map $d\phi : \mathfrak{sl}_2(\Lambda) \to \mathfrak{g}_\Lambda$ given by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto Y$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H$ is an injective Lie algebra homomorphism. To check that $d\phi$ is a homomorphism, one only needs to see that $[X,Y] = H$, $[H,X] = 2X$ and $[H,Y] = -2Y$. If the characteristic of Λ is not 2, injectivity is immediate.

Proposition 8. *Let F be a field of characteristic 0, and let (X,Y,H) be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}_F$. Then there is a unique homomorphism $\phi : \mathrm{SL}_{2/F} \to G_{/F}$ whose tangent map is the Lie algebra homomorphism $d\phi$ associated with the triple (X,Y,H).*

Proof. By [M02, Prop. 7.1], there is a homomorphism $t \mapsto \exp(tX) : \mathbf{G}_{a/F} \to G_{/F}$ for which $\rho(\exp(tX)) = \exp(d\rho(tX))$ for every rational representation ρ of $G_{/F}$.

Let (ρ, V) be a faithful F-representation of $G_{/F}$. Then $d\rho$ restricts to a representation of the $\mathfrak{sl}_2$ triple. The Chevalley group construction [Ste68] applied to the Lie algebra $\mathfrak{sl}_{2/F}$ and the representation $(d\rho, V)$ gives a homomorphism $\phi : \mathrm{SL}_{2/F} \to \mathrm{GL}(V)$ which maps the upper triangular subgroup to the image of $t \mapsto \exp(d\rho(tX))$ and the lower triangular subgroup to the image of $s \mapsto \exp(d\rho(sY))$. Now [Bor91, Prop. 6.12] implies that ϕ takes values in $G_{/F}$ and is unique. It is clear by construction that $d\phi$ has the desired form. □

3. The main result

In this section, we suppose that k is an algebraic closure of the finite field $\mathbf{F}_p$. The split reductive group scheme $G_{/\mathbf{Z}}$ is as in the previous section; we now suppose that $G_{/\mathbf{Z}}$ is (split) *semisimple* and *simply connected.* In particular, $G = G_{/k}$ is simply connected. Note that we reserve the undecorated notations G, $\mathfrak{g}$, etc. for the objects over k. We suppose p to be good for G.

Fix an algebraic closure $E = \overline{\mathbf{Q}}$ of the rational field; in what follows, we regard all finite extensions of $\mathbf{Q}$ as subextensions of $E/\mathbf{Q}$. For F a finite extension of $\mathbf{Q}$ and $\mathscr{A} \subset F$ a valuation ring whose residue field has characteristic p, an element $X \in \mathfrak{g}_{\mathscr{A}}$ determines elements $X_F = X \otimes 1_F \in \mathfrak{g}_F$ $X_E \in \mathfrak{g}_E$, and $X_k \in \mathfrak{g} = \mathfrak{g}_k$ (note that X_k actually depends on the embedding of the residue field of $\mathscr{A}$ in k; the particular choice is not important).

3.1. $\mathfrak{sl}_2$**-triples over integers.** We require the following result due to Spaltenstein. The simple connectivity hypothesis is unnecessary when G has no simple factors of type A_n. Some hypothesis is necessary, though, since the conclusion of the following lemma is not valid for example when $p = 2$ and G is the adjoint group $\mathrm{PGL}_2(k)$.

Lemma 9. *If $X \in \mathfrak{g}$ is a distinguished nilpotent element, then $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathfrak{p}$, where $\mathfrak{p}$ is the canonical parabolic subalgebra attached to X.*

Proof. Since G is simply connected, it is a direct product of simply connected, quasi-simple groups $G \simeq G_1 \times \cdots \times G_r$. For $i = 1, \ldots, r$, let $p_i : G \to G_i$ be the projection. Then $X_i = dp_i(X)$ is distinguished in $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ for each i. Since $\mathfrak{c}_{\mathfrak{g}}(X) = \bigoplus_i \mathfrak{c}_{\mathfrak{g}_i}(X_i)$, and since $\mathfrak{p} = \bigoplus_i \mathfrak{p} \cap \mathfrak{g}_i$, it suffices to assume that G is quasi-simple.

If the root system of G is not of type A_n, the assertion follows from the main result of Spaltenstein in [Spa84]. Otherwise, $G \simeq \mathrm{SL}_n(k)$, and X is a regular nilpotent element in $\mathfrak{g} = \mathfrak{sl}_n(k)$. In this case, $\mathfrak{p} = \mathfrak{b}$ is the Lie algebra of a Borel subgroup, and the claim follows from a direct computation. □

Lemma 10. *Let $\mathfrak{p}_{\mathbf{Z}} \subset \mathfrak{g}_{\mathbf{Z}}$ be a distinguished standard parabolic, and $\tau \in X_*(T_{/\mathbf{Z}})$ the corresponding co-character as in (2). Suppose that a Richardson element $X \in \mathfrak{u}_k$ satisfies $X^{[p]} = 0$.*

1. *There is a finite field extension $F \supset \mathbf{Q}$, a valuation ring $\mathscr{A} \subset F$ (whose residue field we embed in k), and an element $X \in \mathfrak{g}_{\mathscr{A}}(2)$ such that*
   - (a) *$X_k \in \mathfrak{g}(2)$ is a Richardson element for $\mathfrak{p}$,*
   - (b) *$X_E \in \mathfrak{g}_E(2)$ is a Richardson element for $\mathfrak{p}_E$.*
2. *Put $H = d\tau(1) \in \mathfrak{h}_{\mathbf{Z}} \subset \mathfrak{g}_{\mathbf{Z}}(0)$. There is a unique element $Y \in \mathfrak{g}_{\mathscr{A}}(-2)$ such that (X, Y, H) is an $\mathfrak{sl}_2$-triple over $\mathscr{A}$.*

Proof. The assertion (1) is elementary; a proof is written down in [M02, Lemma 5.2]. Since $H = d\tau(1)$, (2) will follow provided that we find $Y \in \mathfrak{g}_{\mathscr{A}}(-2)$ with $[X, Y] = H$. Since $\mathfrak{p}$ is distinguished and $G_{/\mathbf{Z}}$ is semisimple, we have

$$\mathrm{rank}_{\mathbf{Z}}\ \mathfrak{g}_{\mathbf{Z}}(-2) = \mathrm{rank}_{\mathbf{Z}}\ \mathfrak{g}_{\mathbf{Z}}(0).$$

Lemma 9 shows that the centralizer in $\mathfrak{g}$ of X_k is contained in $\mathfrak{p}$. This implies that $\mathrm{ad}(X_k) : \mathfrak{g}(-2) \to \mathfrak{g}(0)$ is injective, and is therefore a linear isomorphism. Thus $\mathrm{ad}(X) : \mathfrak{g}_{\mathscr{A}}(-2) \to \mathfrak{g}_{\mathscr{A}}(0)$ is also bijective and (2) follows. □

3.2. **Exponential isomorphism.**

Lemma 11. *Let $\mathfrak{p}_{\mathbf{Z}}$ be a distinguished standard parabolic subalgebra of $\mathfrak{g}_{\mathbf{Z}}$, and let $\mathfrak{u}_{\mathbf{Z}}$ be as in 2.2. Suppose that a Richardson element $X \in \mathfrak{u}_k$ satisfies $X^{[p]} = 0$. Then the exponential isomorphism* $\exp : \mathfrak{u}_{\mathbf{Q}} \to U_{\mathbf{Q}}$ *is defined over $\mathbf{Z}_{(p)}$.*

Proof. Recall that $\mathfrak{u}_{\mathbf{Z}}$ is the Lie algebra of the group scheme $U_{\mathbf{Z}}$; $U_{\mathbf{Q}}$ is obtained by base change. Since $X^{[p]} = 0$, [M02, Theorem 5.4] shows that that $\mathfrak{g}_{\mathbf{Z}}(i) = 0$ for all $i \geq 2p$ (for the grading induced by the cocharacter τ in (2)). It now follows [2] that the nilpotence class of the Lie algebra $\mathfrak{u}_k$ is $< p$. The lemma now follows from [Sei00, Prop. 5.1]. □

Lemma 12. *Let F be an arbitrary subfield of k, and suppose that G is defined (not necessarily split) over F. Let $\mathfrak{p}$ be a parabolic subalgebra defined over F, and*

[2] In [M02, §4.4], $n(P)$ is defined as the least $n \geq 0$ with $\mathfrak{g}(2n) = 0$ for the grading induced by τ as in (2). Write $c(\mathfrak{u})$ for the nilpotence class of $\mathfrak{u}$, and $c(U)$ for that of U. [M02, Prop. 4.4] erroneously asserts that $c(\mathfrak{u})$, $c(U)$ and $n(P)$ coincide; in fact the given proof shows that $n(P) - 1 = c(\mathfrak{u}) = c(U)$. In the above situation, we therefore see that $p \geq n(P) > c(\mathfrak{u})$ whence the claim. This error in [M02, Prop. 4.4] led to a flawed statement of [M02, Theorem 1.1]; a correct statement is obtained by taking $n(P)$ to be $c(\mathfrak{u}) + 1$ (rather than $c(\mathfrak{u})$).

suppose that a Richardson element X of its nilradical $\mathfrak{u}$ satisfies $X^{[p]} = 0$. Then there is a unique P-equivariant isomorphism of varieties $\exp : \mathfrak{u} \to U$ *whose tangent map is the identity. Moreover,* $\exp$ *is defined over F.*

Proof. As in the proof of the previous lemma, the hypothesis guarantees that the nilpotence class of $\mathfrak{u}$ is $< p$; the result then follows from [Sei00, Proposition 5.2]. □

3.3. **The main theorem.** It is:

Theorem 13. *Let $\mathfrak{p}_{\mathbf{Z}} \subset \mathfrak{g}_{\mathbf{Z}}$ be a distinguished standard parabolic, and $\tau \in X_*(T_{/\mathbf{Z}})$ the corresponding co-character as in (2). Suppose that a Richardson element $Y \in \mathfrak{u}_k$ satisfies $Y^{[p]} = 0$.*

Choose the number field F, valuation ring $\mathscr{A}$, and element $X \in \mathfrak{g}_{\mathscr{A}}(2)$ as in Lemma 10(1), and let (X, Y, H) be an $\mathfrak{sl}_2$-triple over $\mathscr{A}$ as in that lemma.

(1) *Let $\phi_{/F} : \mathrm{SL}_{2/F} \to G_{/F}$ be the homomorphism determined by the $\mathfrak{sl}_2$ triple (X_F, Y_F, H_F) as in Proposition 8. Then $\phi_{/F}$ is defined over $\mathscr{A}$; i.e. there is a homomorphism of group schemes $\phi_{/\mathscr{A}} : \mathrm{SL}_{2/\mathscr{A}} \to G_{/\mathscr{A}}$ from which $\phi_{/F}$ arises by scalar extension.*
(2) *If $\phi_{/k} : \mathrm{SL}_{2/k} \to G$ denotes the map obtained by base change from $\phi_{/\mathscr{A}}$, then:*
   (a) *The image of $\phi_{/k}$ meets the open P-orbit on U, where U is the unipotent radical of the parabolic subgroup P corresponding to $\mathfrak{p}$.*
   (b) *The Richardson element X_k is in the image of $d\phi_{/k}$.*
   (c) *The restriction of $\phi_{/k}$ to a suitable maximal torus of $\mathrm{SL}_{2/k}$ coincides with the co-character τ; it is a co-character associated to X_k.*

Proof. Lemma 11 implies that the exponential isomorphism $\exp : \mathfrak{u}_{\mathbf{Q}} \to U_{\mathbf{Q}}$ is defined over $\mathbf{Z}_{(p)}$ and hence over $\mathscr{A}$. Applying this lemma to the opposite parabolic, the isomorphism $\exp : \mathfrak{u}^-_F \to U^-_{/F}$ is defined over $\mathscr{A}$ as well.

Consider the subgroup schemes of $\mathrm{SL}_2 = \mathrm{SL}_{2/\mathbf{Z}}$:

$$U_1^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \simeq \mathbf{G}_{a/\mathbf{Z}}, \quad T_1 = \mathbf{G}_{m/\mathbf{Z}}, \quad U_1 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \simeq \mathbf{G}_{a/\mathbf{Z}}.$$

The "big cell" of SL_2 is the subscheme $\Omega = U_1^- \cdot T_1 \cdot U_1$; the product map defines an isomorphism $U_1^- \times T_1 \times U_1 \to \Omega$ of schemes $/\mathbf{Z}$.

If $\phi_{/F} : \mathrm{SL}_{2/F} \to G_{/F}$ is as in (1), then the restriction of $\phi_{/F}$ to $\Omega_{/F}$ is given by $(s, t, u) \mapsto \exp(sY_F) \cdot \tau_{/F}(t) \cdot \exp(uX_F)$. Since τ and the exponential maps are defined over $\mathscr{A}$, it follows that the restriction of $\phi_{/F}$ to $\Omega_{/F}$ is defined over $\mathscr{A}$. The proof of [Ser96, Prop. 2] now implies that $\phi_{/F}$ is defined over $\mathscr{A}$ which settles (1). (The argument of *loc. cit.* uses that $\mathrm{SL}_{2/\mathscr{A}}$ is covered by $\Omega_{/\mathscr{A}}$ and $w\Omega_{/\mathscr{A}}$, for a suitable $w \in \mathrm{SL}_2(\mathbf{Z})$).

To prove (2), note first that by Lemma 12, $\exp : \mathfrak{u} \to U$ is P-equivariant. Since X_k is in the dense P orbit on $\mathfrak{u}$, $\exp(X_k)$ is in the dense P orbit on U; since $\exp(X_k) = \phi_{/k}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$, (a) now follows. Since by Lemma 12 the tangent map to $\exp$ is the identity, (b) holds. (c) follows by construction. □

4. Why Theorem 13 implies Theorems 1 and 2

We return to the assumptions of the introduction; thus k is algebraically closed of characteristic $p > 0$ (but not necessarily an algebraic closure of a finite field), G is reductive, and p is good for G.

Since Theorem 2 is a stronger form of Theorem 1; we just prove this latter result using Theorem 13.

Proof of Theorem 2. First, we may replace the reductive group G by its derived group, so that we may suppose G to be semisimple. Let $\pi : \hat{G} \to G$ be the simply connected covering group of G. According to Lemma 27 (in the appendix), the restrictions $\pi_{|\hat{\mathcal{U}}} : \hat{\mathcal{U}} \to \mathcal{U}$ and $d\pi_{|\hat{\mathcal{N}}} : \hat{\mathcal{N}} \to \mathcal{N}$ are homeomorphisms. If $u' = \pi^{-1}(u)$ and if $\phi : \mathrm{SL}_{2/k} \to \tilde{G}$ is a sub-principal homomorphism with u' in its image, it follows from Remark 5 that $\pi \circ \phi$ is a sub-principal homomorphism with u in its image. Thus we may suppose that G is simply connected.

It suffices to prove the theorem in the case where k is an algebraic closure of a finite field. Indeed, let $k_0 \subset k$ denote the algebraic closure of the prime field $\mathbf{F}_p$. Then by [M02, Cor. 7.3] each nilpotent orbit in $\mathfrak{g}_k$ has a point rational over k_0.

Since p is good, there is a G-equivariant homeomorphism $\varepsilon : \mathcal{N} \to \mathcal{U}$; see Proposition 29 of the appendix. Moreover, by Remark 30 (in the appendix), ε restricts to a G-equivariant homeomorphism $\mathcal{N}_{/k_0} \to \mathcal{U}_{/k_0}$. This shows that also each unipotent class has a point rational over k_0. Thus $u = gu'g^{-1}$ for some u' rational over k_0, where g is over k. The theorem for k_0 and u' yields a suitable homomorphism $\phi : SL_{2/k_0} \to G_{/k_0}$, and $\mathrm{Int}(g) \circ \phi$ then works for k and u.

Now assume that $k = k_o$, and that u is a distinguished unipotent element. Let P be the canonical parabolic subgroup attached to u. Replacing u and P by a G-conjugate, we may suppose that $\mathrm{Lie}(P) = \mathfrak{p}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$ where $\mathfrak{p}_{\mathbf{Z}}$ is a distinguished standard parabolic of $\mathfrak{g}_{\mathbf{Z}}$.

Since u has order p, [M02, Theorem 1] shows that a Richardson element X in $\mathrm{Lie}(U)$ satisfies $X^{[p]} = 0$. Theorem 13 now gives us a sub-principal homomorphism $\phi : \mathrm{SL}_{2/k} \to G_{/k}$ whose image meets the dense P orbit on U. Replacing ϕ by $\mathrm{Int}(g) \circ \phi$ for a suitable $g \in P$, the proof is complete for distinguished u.

When u is not distinguished, it is distinguished in a proper Levi subgroup L. We may then apply the result in the distinguished case to L; Remark 4 shows that the homomorphism so obtained has the desired properties. □

5. Rationality properties

Let F be a ground field, suppose that $G = G_{/F}$ is a reductive group over F.

5.1. **Some rational Levi and parabolic subgroups.** We begin with a lemma:

Lemma 14. *Let P be a parabolic subgroup of G, and suppose that some maximal torus of P is defined and split over F. Then P is itself defined over F.*

Proof. Note that the our assumption means that G is F-split. Let $T_o \leq B_o$ be the "standard" maximal torus and Borel subgroup of the split group G. Thus T_o is F-split, and representatives for the cosets in the Weyl group $W = N_G(T_o)/T_o$ may be chosen rational over F.

By [Spr98, Theorem 14.4.3] one knows that any two F-split maximal tori of G are conjugate by an element of $G(F)$. Thus we may as well suppose that P contains T_o.

Choose a Borel subgroup B of P containing T_o. Since B is also a Borel subgroup of G, and since all Borel subgroups of G containing T_o are conjugate by $N_G(T_o)$, we may replace P by a $G(F)$ conjugate and suppose that P contains B_o. Now the lemma is immediate, since each of the parabolic subgroups of G containing the standard Borel subgroup B_o are defined over F. □

Recall [J, §2.9] that when G is semisimple and the characteristic p of F is *very good* for G, then the orbit of each nilpotent and unipotent element is separable (if the irreducible root system is different from A_r, then "very good" means the same as "good", while p is "very good" for type A_r provided that $r \not\equiv -1 \pmod p$). When G is reductive, we refer to *loc. cit.* for a discussion of the separability of nilpotent orbits.

Theorem 15. *Let $X \in \mathfrak{g}(F)$ be an F-rational nilpotent element, and suppose either that F is perfect, or that the G-orbit of X is separable.*

1. *Then X is distinguished in the Lie algebra of a Levi subgroup which is defined over F.*
2. *The canonical parabolic subgroup attached to X is defined over F.*

Proof. For the first assertion, let $C = C_G(X)$ be the centralizer of X. Under our assumptions, either F is perfect, or the orbit map for X is separable. According to [Spr98, Prop. 12.1.2], the group C is then defined over F.

Let $T \leq C$ be a maximal torus which is defined over F (such a torus exists by [Spr98, Theorem 13.3.6 and Remark 13.3.7]). Then $L = C_G(T)$ is a Levi subgroup of G whose Lie algebra contains X; see [DM91, Prop. 1.22] and [Spr98, Corollary 5.4.7]. The Levi subgroup L is F-rational by [Spr98, Prop. 13.3.1]. Since T is central in L, we see that the connected center of L is a maximal torus of $C_L(X)$ hence that X is distinguished in $\mathrm{Lie}(L)$; this proves (1).

For (2), let $\mathcal{O} = \mathrm{Ad}(G)X \subset \mathfrak{g}$, let P be the canonical parabolic subgroup associated to X, and let $\mathfrak{p} = \mathrm{Lie}(P)$; see Proposition 7. We will show that P (and hence also $\mathfrak{p}$) is defined over F.

Fix an algebraic closure $\overline{F}$ of F, let F_s be the separable closure of F in $\overline{F}$, and let Γ be the Galois group of F_s/F. Since X is F-rational, it is stable under the action of Γ; since P is canonically attached to X, it is also stable under the action of Γ. If P is defined over F_s, (2) now follows by [Spr98, Proposition 11.2.8(i)]. This completes the proof in case F is perfect. In the case where F is not perfect, this shows that *we may now suppose F to be separably closed*, and we may moreover suppose that the orbit map $G \to \mathcal{O}$ is separable.

Let $\mathscr{P}$ be the variety of parabolic subgroups conjugate to P, and let $P_0 \in \mathscr{P}$ denote the standard parabolic which is conjugate (via G) to P. Since G is split over F, P_0 is over F. We identify $\mathscr{P}$ with G/P_0 as F-varieties. We have to show that P is conjugate to P_0 via $G(F)$.

Since the quotient map $G \to G/P_o$ is defined over F and has "local sections" (see [Spr98, Lemma 8.5.2]) we may define the fiber space

$$Y = G \times^{P_o} \mathfrak{u}_o$$

as in [Spr98, Lemma 5.5.8], where $\mathfrak{u}_o$ is the nilradical of $\mathrm{Lie}(P_o)$. Y is an F-variety and is defined as a quotient of $G \times \mathfrak{u}_o$. Since the local sections of $G \to G/P_o$ are defined over F, a point in Y is F-rational if and only if it is represented by an F-rational pair $(g, Z) \in G \times \mathfrak{u}_o$. Let $p_1 : Y \to \mathscr{P}$ denote the morphism induced by $(g, Z) \mapsto \mathrm{Int}(g)P_o$, and let $p_2 : Y \to \overline{\mathcal{O}} \subset \mathcal{N}$ be the morphism induced by $(g, Z) \mapsto \mathrm{Ad}(g)Z$. Then $p_1 : Y \to \mathscr{P}$ is a vector bundle, hence Y is a smooth F-variety. Moreover, p_1 and p_2 are defined over F, and are equivariant (for the obvious action of G on Y). Let $\mathcal{U} = p_2^{-1}(\mathcal{O})$.

Now suppose that X is distinguished, so that X is a Richardson element for P. Thus $\overline{\mathrm{Ad}(P)X} = \mathfrak{u}$. Proposition 7 implies that $C_G(X) = C_P(X)$. Since the orbit map $G \to \mathcal{O}$ is separable, [J, Lemma §8.8] implies that

(3) $\qquad p_2$ restricts to an isomorphism $\mathcal{U} \xrightarrow{\sim} \mathcal{O}$ of varieties.

Consider $\tilde{X} = p_2^{-1}(X) \in \mathcal{U}$. Then $\tilde{X}$ is a simple point of Y (since Y is smooth). Since $X \in \mathcal{U}$, (3) implies that $dp_2 : T_{\tilde{X}}Y \to T_X\overline{\mathcal{O}}$ is an isomorphism. We now apply the condition [3] of [Spr98, Corollary 11.2.14] to see that the $\tilde{X}$ is rational over F. Thus $\tilde{X}$ is represented by a pair (g, Z) where $g \in G(F)$ and $Z \in \mathfrak{u}_o(F)$, so that $P = gP_og^{-1}$ is defined over F. This proves (2) in case X is distinguished.

If X is any nilpotent, then by (1) X is distinguished nilpotent in the Lie algebra of a Levi subgroup L which is defined over F. The canonical parabolic Q of X in the Levi subgroup L is then defined over F by the previous remarks. Recall that we suppose F to be separably closed; thus Q is split over F. The canonical parabolic subgroup P of X in the original group G contains Q. It follows that P contains a split maximal torus of G, so P is defined over F by Lemma 14. ☐

Theorem 16. *Let $u \in G(F)$ be an F-rational unipotent element, and suppose either that F is perfect, or that the G-orbit of u is separable.*

(1) *u is distinguished in a Levi subgroup which is defined over F.*
(2) *The canonical parabolic subgroup associated to u is defined over F.*

Proof. Part (1) is proved *mutatis mutandum* as in part (1) of the previous theorem. For part (2) in the distinguished case, one must instead replace the vector bundle Y by the "affine-space bundle" $G \times^{P_o} U_o$ over $\mathscr{P}$. The remainder of the argument is the same. ☐

Remark 17. In the case where $F = \mathbf{F}_q$ is a finite field of order q, we can give a different proof of part (2) of Theorem 15 which shows moreover that the *grading* of 2.2 is $\mathbf{F}_q$-rational. Indeed, let $\phi : \mathbf{G}_m \to G$ be a cocharacter associated to X. Then ϕ determines a subgroup $T_\phi = \{(t, \phi(t)) \mid t \in \mathbf{G}_m\} \leq \mathbf{G}_m \times G$. Since X is $\mathbf{F}_q$-rational, the variety of all such subgroups T_ϕ (where ϕ ranges over all cocharacters associated to X) is stable by the action of the geometric Frobenius endomorphism, and is thus rational over $\mathbf{F}_q$ (see Lemma 22 and the proof of Theorem 23 below). Moreover, this variety is a homogeneous space for the connected group $C_G^o(X)$ by Proposition 6. Thus an application of Lang's Theorem [DM91, Cor. 3.12] shows that there is an $\mathbf{F}_q$ rational point $T_{\phi'}$. The rationality of $T_{\phi'}$ is equivalent to that of ϕ'; thus the weight spaces of ϕ' on $\mathfrak{g}$ are $\mathbf{F}_q$-rational.

Remark 18. Let F be an arbitrary ground field, and X a rational nilpotent element. Despite the rationality of the canonical parabolic associated with X, I do not know if there is always a cocharacter associated with X which is defined over F. This is so for finite fields by the previous remark.

[3]There is a typographic error in the statement of Corollary 11.2.14 in [Spr98]: the condition on $d\phi_x$ should read "...the tangent map $d\phi_x : T_xX \to T_yY$ is surjective...".

5.2. **Conjugacy of nice homomorphisms.** In this section, G is a reductive group over the algebraically closed field k. Moreover, we suppose that G is *simply connected*. Fix a distinguished parabolic subgroup P of G, and let $\varepsilon : \mathfrak{u} \to U$ be a P-equivariant homeomorphism, where U is the unipotent radical of P and $\mathfrak{u}$ is its Lie algebra (in good characteristic, such a homeomorphism always exists; see §6 below.)

Fix a Richardson element $u \in U$, and let $X = \varepsilon^{-1}(u)$. We shall say that a homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ is *nice* for u with respect to ε if the following property is satisfied:

$$\phi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = u \quad \text{and} \quad d\phi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = X. \tag{4}$$

For a nice homomorphism ϕ, let ψ be the co-character given by

$$\psi(t) = \phi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}), \quad t \in k^\times; \tag{5}$$

since X is distinguished, it is immediate that ψ is associated to X.

Proposition 19. *Any two homomorphisms $\mathrm{SL}_{2/k} \to G$ which are nice for u with respect to ε are conjugate by an element of $C^o_G(u)$.*

Proof. For $i = 1, 2$ let ϕ_i be nice homomorphisms for u with respect to ε, and let ψ_i be the corresponding characters as in (5). According to [J, Lemma 5.3], there is an element $g \in C^o_G(u) = C^o_G(X)$ with $\psi_2 = \mathrm{Int}(g) \circ \psi_1$. Replacing ϕ_1 by $\mathrm{Int}(g) \circ \phi_1$, we may suppose that $\psi_1 = \psi_2$. The proposition is now a consequence of the lemma that follows. □

Lemma 20. *Let $\phi_i : \mathrm{SL}_{2/k} \to G$, $i = 1, 2$, be nice homomorphisms for u with respect to ε, and let ψ_i be the corresponding cocharacters as in* (5)*. If $\psi_1 = \psi_2$, then $\phi_1 = \phi_2$.*

Proof. Let $\Omega \subset \mathrm{SL}_{2/k}$ be the big cell $\Omega = U_1^- T_1 U_1$ as in the proof of Theorem 13. Since Ω is a dense subset of $\mathrm{SL}_{2/k}$, it suffices to show that the restrictions of ϕ_1 and ϕ_2 to Ω coincide.

For $s \in k$, one has

$$\phi_i(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}) = \mathrm{Int}(\psi_i(s^{1/2}))u;$$

since $\psi_1 = \psi_2$, it follows that the restrictions of the ϕ_i to U_1 coincide.

Let $\mathfrak{g}(i)$ the graded components of $\mathfrak{g}$ with respect to ψ, and let $H = d\psi(1) \in \mathfrak{g}(0)$. Since G is simply connected, Lemma 10 shows that there is a unique $Y \in \mathfrak{g}(-2)$ such that (X, H, Y) is an $\mathfrak{sl}_2$ triple over k; it follows that $d\phi_i(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = Y$ for $i = 1, 2$. In particular, $d\phi_1 = d\phi_2$.

We may find $w \in \mathrm{SL}_{2/k}$ with

$$\mathrm{Ad}(w) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{Int}(w) \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \quad \text{for all } s \in k.$$

Since $d\phi_1 = d\phi_2$, we find that $\phi_1(w)^{-1}\phi_2(w) \in C_G(X) = C_G(u)$. It then follows that $\phi_1(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = \phi_2(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$. Arguing as before, one sees that the restrictions of the ϕ_i to U_1^- coincide, and the lemma is proved. □

Remark 21. Let $u \in G$ be a distinguished unipotent element of order p, and suppose that u is rational over a ground field F. We make the same hypothesis as in Theorems 15 and 16; thus p is a good prime, and either the G-orbit of u is separable, or F is perfect. Let U be the unipotent radical of the canonical parabolic subgroup P associated with u (recall by Theorem 16 that P and hence U are defined over F). By Lemma 12 there is a unique P-equivariant isomorphism $\exp : \mathrm{Lie}(U) \to U$ whose tangent map is the identity; moreover, exp is defined over F. Since G is simply connected, our proof of Theorem 2 via Theorem 13 shows that there is a sub-principal homomorphism $\phi : \mathrm{SL}_{2/k} \to G$ which is nice for u with respect to exp. Actually, it is not necessary to assume simple connectivity: if $\pi : \hat{G} \to G$ is the simply connected covering group, then π restricts to an isomorphism $\pi_{|\hat{U}} : \hat{U} \to U$ where $\hat{U} = \pi^{-1}(U)$. By the unicity, we have $\exp \circ d\pi_{|\hat{\mathfrak{u}}} = \pi_{|\hat{U}} \circ \hat{\exp}$, where $\hat{\exp}$ denotes the corresponding exponential for $\hat{U}$. So if $\hat{\phi} : \mathrm{SL}_{2/k} \to \hat{G}$ is nice for $\pi^{-1}(u)$ with respect to $\hat{\exp}$, then $\phi = \pi \circ \hat{\phi}$ is nice for u with respect to exp.

Note that there is no *a priori* reason that ϕ should be defined over F.

5.3. **Finite fields.** Suppose now that k is an algebraic closure of the finite field $\mathbf{F}_q$ with q elements and characteristic p. Let V be an affine variety over k. Recall that there is a dictionary between $\mathbf{F}_q$-structures on V and certain morphisms $F : V \to V$ (for details consult e.g. [DM91, Chapter 3]). Indeed, an $\mathbf{F}_q$-structure on V is a finitely generated $\mathbf{F}_q$-subalgebra $A_0 = \mathbf{F}_q[V] \subset A = k[V]$ with the property that the natural map $A_o \otimes_{\mathbf{F}_q} k \to A$ is an isomorphism. The co-morphism $F^* : A \to A$ is then given by $f \otimes \alpha \mapsto f^q \otimes \alpha$ for $f \in A_o$ and $\alpha \in k$. Conversely, F determines A_o as $\{f \in A \mid F^* f = f^q\}$. Note that $F^* : A \to A^q$ is surjective, and that [DM91, Prop. 3.3(i)] gives necessary and sufficient conditions under which a surjective algebra map $A \to A^q$ determines an $\mathbf{F}_q$-structure on V. The map F is called the *geometric Frobenius* endomorphism of V.

Lemma 22. *Let H be a linear algebraic k-group defined over $\mathbf{F}_q$, and let F be the corresponding Frobenius endomorphism of H.*

(1) *There is a unique q-semilinear automorphism φ of $\mathfrak{h} = \mathrm{Lie}(H)$ such that the $\mathbf{F}_q$-space of φ-fixed-points $\mathfrak{h}^{\varphi}$ identifies with the H-invariant $\mathbf{F}_q$-derivations of $\mathbf{F}_q[H]$.*
(2) *If $B \leq H$ is a closed subgroup,* $\mathrm{Lie}(F(B)) = \varphi\,\mathrm{Lie}(B)$.

Proof. First recall that a $\mathbf{F}_p$-linear automorphism φ of a k-vector space V is q-semilinear if $\varphi(\lambda v) = \lambda^q \varphi(v)$ for each $v \in V$ and $\lambda \in k$.

Let $A_0 = \mathbf{F}_q[H] \subset A = k[H]$. Recall that the *arithmetic Frobenius* map φ_a associated to the give $\mathbf{F}_q$-structure is the q-semilinear automorphism of A satisfying $\phi_a(f \otimes \alpha) = f \otimes \alpha^q$ for $f \in A_0$, $\alpha \in k$.

Now $\mathfrak{h}$ is the Lie algebra of H-invariant derivations of A. There is a natural map $\mathrm{Der}_{\mathbf{F}_q}(A_0) \to \mathrm{Der}_k(A)$. The H-invariant derivations in the image of this map form an $\mathbf{F}_q$-subspace $\mathfrak{h}_0$ of $\mathfrak{h}$; moreover, the natural map $\mathfrak{h}_0 \otimes_{\mathbf{F}_q} k \to \mathfrak{h}$ is an isomorphism. Now take for φ the map satisfying $\varphi(X \otimes \alpha) = X \otimes \alpha^q$ (for each $X \in \mathfrak{h}_0$ and $\alpha \in k$). Then $\mathfrak{h}_0 = \mathfrak{h}^{\phi}$ and (1) is clear. If $X \in \mathfrak{h}$ is regarded as a derivation, and $f \in A$, then $\varphi^{-1}(X)(f) = \varphi_a^{-1}(X(\varphi_a(f)))$.

Let $I = \mathcal{I}(B) \lhd A$ be the defining ideal of the closed subgroup B, and let $J = \mathcal{I}(FB) \lhd A$ that of FB. Since $F^* \circ \phi_a(f) = \phi_a \circ F^*(f) = f^q$ for each $f \in A$, one readily checks that $J = \phi_a(I)$. Thus,

$$\begin{aligned}\operatorname{Lie}(F(B)) &= \{X \in \mathfrak{h} \mid X(f) \in J \ \forall f \in J\} \\ &= \{X \in \mathfrak{h} \mid \varphi_a^{-1}(X(\varphi_a(h))) \in I \ \forall h \in I\} \\ &= \{X \in \mathfrak{h} \mid \varphi^{-1}(X)(h) = 0 \ \forall h \in I\} \\ &= \varphi \operatorname{Lie}(B);\end{aligned}$$

this proves (2). □

We now suppose that the connected reductive group G is defined over $\mathbf{F}_q$ and that p is good for G. Denote by F the corresponding Frobenius endomorphism of G, and by φ the q-semilinear automorphism of $\mathfrak{g}$ as in the lemma. Also, let F_o be the Frobenius endomorphism of $\mathrm{SL}_{2/k}$ (and φ_0 the q-semilinear automorphism of $\mathfrak{sl}_2(k)$) for the standard $\mathbf{F}_q$-structure.

Theorem 23. *Let $u \in G$ be an $\mathbf{F}_q$-rational unipotent element of order p. Then there is a sub-principal homomorphism $\psi : \mathrm{SL}_{2/\mathbf{F}_q} \to G_{/\mathbf{F}_q}$ defined over $\mathbf{F}_q$ whose image contains u.*

Proof. As in the proof of Theorem 2, we may suppose that G is simply connected (note that the simply connected covering $\pi : \hat{G} \to G$ is defined over $\mathbf{F}_q$. In fact, for *any* ground field F, the simply connected cover of an F-reductive group is again over F; see [Spr98, Lemma 16.2.4].)

By Theorem 16(1), we may suppose that u is distinguished. Let P be the canonical parabolic associated to u, and let U be its unipotent radical (by Theorem 16 these subgroups are defined over $\mathbf{F}_q$). By Remark 21, there is a sub-principal homomorphism ψ which is nice for u with respect to exp, where $\exp : \operatorname{Lie}(U) \to U$ is the exponential $\mathbf{F}_q$-isomorphism of Lemma 12. For any sub-principal homomorphism ψ nice for u with respect to exp, we get a subgroup

$$\Gamma = \Gamma_\psi = \{(g, \psi(g)) \mid g \in \mathrm{SL}_{2/k}\} \leq \mathrm{SL}_{2/k} \times G$$

satisfying:

(1) Γ contains $u' = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u)$.
(2) $\operatorname{Lie}(\Gamma)$ contains $X' = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X)$ (where $u = \exp(X)$).
(3) The restriction of the first projection p_1 to Γ is an isomorphism (of algebraic groups).

On the other hand, if $\Gamma \leq \mathrm{SL}_{2/k} \times G$ is any subgroup satisfying (1), (2), and (3), then Γ is the graph of a unique sub-principal homomorphism $\psi : \mathrm{SL}_{2/k} \to G$ which is nice for u with respect to exp.

Since G is simply connected, it follows from Proposition 19 that the variety $\mathcal{V}$ of all subgroups Γ satisfying (1), (2), and (3) is a homogeneous space for the connected group $C_G^o(u)$.

We now claim that the variety $\mathcal{V}$ is stable by the Frobenius $F_1 = F_o \times F$ of $\mathrm{SL}_{2/k} \times G$. Let Γ in $\mathcal{V}$; we verify that (1)–(3) hold for $F_1\Gamma$. Since u' is clearly F_1 stable, (1) holds. Since exp is defined over $\mathbf{F}_q$, $X' \in \mathfrak{sl}_2(k) \oplus \mathfrak{g}$ is fixed by $\varphi_1 = \varphi_0 \oplus \varphi$. According to the lemma, the Lie algebra of $F_1\Gamma$ is $\varphi_1 \operatorname{Lie}(\Gamma)$; it follows that $X' \in \operatorname{Lie}(F_1\Gamma)$ so that (2) holds.

To verify (3), note that the restriction of p_1 to $F_1\Gamma$ is evidently bijective (since F_1 itself is bijective). It remains to see that the restriction of the differential dp_1 to $\mathrm{Lie}(F_1\Gamma)$ is bijective; that follows immediately from the equality $\mathrm{Lie}(F_1\Gamma) = \varphi\,\mathrm{Lie}(\Gamma)$ proved in the lemma.

We have now verified that $\mathcal{V}$ is F_1 stable. Since this variety is a homogeneous space for the connected group $C^o_G(u)$, an application of Lang's Theorem [DM91, Cor. 3.12] yields a point $\Gamma \in \mathcal{V}$ fixed by F_1. The homomorphism ψ whose graph is Γ then has the desired properties. □

Remark 24. The theorem yields in particular a homomorphism $\mathrm{SL}_2(\mathbf{F}_q) \to G(\mathbf{F}_q)$ between the groups of rational points.

Remark 25. It is common in the study of finite simple groups to consider a more general notion of Frobenius endomorphism. Let G be a semisimple group over k. A surjective endomorphism σ of G will be called a Frobenius endomorphism provided the fixed-point group G^σ is finite; such endomorphisms are thoroughly studied by Steinberg in [Ste68a].

Let σ be a Frobenius endomorphism of the semisimple group G, and suppose that $u \in G$ is σ-stable and of order p. It is proved in [PST00, Theorem 5.1] that in case σ is a q-Frobenius endomorphism (for the definition, see [PST00, §1]), there is a σ-stable closed A_1-type subgroup $S \leq G$ containing u. In general, there need be no such subgroup; see [PST00, Lemma 2.1(i) and Lemma 2.2].

We observe that σ is a q-Frobenius endomorphism (in the sense of [PST00]) if and only if it is the geometric Frobenius endomorphism associated to an $\mathbf{F}_q$-structure of G. This observation was made in a slightly different context in 11.6 (p. 76) of Steinberg's paper on Endomorphisms of Linear Algebraic Groups [Ste68a]; note that if σ is a q-Frobenius, the results in [DM91, Chapter 3] (cited above) yield the existence of a suitable $\mathbf{F}_q$-structure, while if σ is not a q-Frobenius, then σ^*A is not equal to $A^{q'}$ for any $q' = p^a$, where $A = k[G]$.

Thus, when G is semisimple in good characteristic, the content of Theorem 23 is roughly that of [PST00, Theorem 5.1]. Note however that on the one hand, the theorem in [PST00, Theorem 5.1] treats also bad primes (the reader is referred there for a precise statement in bad characteristic), while on the other hand, Theorem 23 gives more precise information about the A_1-embedding.

Since the proof of Theorem 23 is achieved essentially through an application of Lang's theorem, and since Lang's theorem remains valid for any Frobenius endomorphism σ, the reader may be curious why Theorem 23 is not valid for an arbitrary σ. The reason is as follows: If $\mathcal{V}$ is the variety appearing in the proof, and if σ is not the geometric Frobenius for an $\mathbf{F}_q$-structure on G, then $\mathcal{V}$ need not be $(F_0 \times \sigma)$-stable (note that there is no analogue of Lemma 22 for σ).

6. Appendix: Comparing the unipotent and nilpotent varieties

Let k be an algebraically closed field with characteristic $p \geq 0$, and G a reductive group over k.

Lemma 26. *Suppose $p > 0$, and consider an algebraic torus T over k. Let $\mathfrak{z} \subset \mathrm{Lie}(T)$ be a p-Lie subalgebra. Then $A \mapsto A^{[p^n]}$ defines a homeomorphism $\mathfrak{z} \to \mathfrak{z}$ for each $n \geq 1$.*

Proof. We first observe that the map $\varphi : \mathbf{A}^m_{/k} \to \mathbf{A}^m_{/k}$ given by

$$(x_1, \dots, x_m) \mapsto (x_1^{p^n}, \dots, x_m^{p^n})$$

is a homeomorphism for all $m, n \geq 1$. Indeed, this map is a morphism of varieties, hence continuous; moreover, it is evidently bijective. That it is a homeomorphism will follow provided that it is open. If g is a regular (i.e. polynomial) function on $\mathbf{A}^m_{/k}$, let $D(g)$ denote the distinguished open subset of $\mathbf{A}^m_{/k}$ defined by it. There is a polynomial function f with $g^{p^n} = \varphi^* f$; thus $\varphi(D(g)) = \varphi(D(g^{p^n})) = D(f)$ is open as desired.

Let $\sigma : \mathrm{Lie}(T) \to \mathrm{Lie}(T)$ be the map $A \mapsto A^{[p]}$; for a subspace $\mathfrak{z} \subset \mathrm{Lie}(T)$, we write $\mathfrak{z}^\sigma = \{A \in \mathfrak{z} \mid A = \sigma(A)\}$. Since $\mathrm{Lie}(T)$ is an Abelian algebra, σ is additive and "semilinear": $\sigma(\alpha A) = \alpha^p \sigma(A)$ for $\alpha \in k$ and $A \in \mathrm{Lie}(T)$. Thus $\mathfrak{z}^\sigma$ is an $\mathbf{F}_p$-vector space.

One knows that the canonical map $\mathrm{Lie}(T)^\sigma \otimes_{\mathbf{F}_p} k \to \mathrm{Lie}(T)$ is an isomorphism (indeed: it suffices to observe that this is true when $T = \mathbf{G}_m$).

We now claim that a k-subspace $\mathfrak{z} \subset \mathrm{Lie}(T)$ is a p-subalgebra if and only if the canonical map $\mathfrak{z}^\sigma \otimes_{\mathbf{F}_p} k \to \mathfrak{z}$ is an isomorphism.

This claim follows from the (apparently) more general statement: suppose that V is a finite dimensional k-vector space, that $\sigma : V \to V$ is a bijective, additive, semilinear map, and that $V^\sigma \otimes_{\mathbf{F}_p} k \to V$ is an isomorphism. Then a k-subspace $W \subset V$ is σ stable if and only if the canonical map $W^\sigma \otimes_{\mathbf{F}_p} k \to W$ is an isomorphism; for this, see the proof of [Bor91, Proposition AG 14.2].

To finish the proof of the lemma, note that the choice of an $\mathbf{F}_p$-basis for $\mathfrak{z}^\sigma$ identifies the map $(A \mapsto \sigma^n(A) = A^{[p^n]}) : \mathfrak{z} \to \mathfrak{z}$ with $\varphi : \mathbf{A}^{\dim \mathfrak{z}}_{/k} \to \mathbf{A}^{\dim \mathfrak{z}}_{/k}$ and the lemma follows. □

In the situation of the lemma, we will write $\Theta_n : \mathfrak{z} \to \mathfrak{z}$ for the inverse of the homeomorphism $A \mapsto A^{[p^n]}$, for $n \geq 1$. Thus $\Theta_n(A)$ is a sort of "p^n-th root" of $A \in \mathfrak{z}$. Note that Θ_n is not a morphism of varieties (since the morphism $A \mapsto A^{[p^n]}$ is purely inseparable of degree p^n).

We denote by $\mathcal{U}(G) = \mathcal{U}$ the variety of unipotent elements in G, and by $\mathcal{N}(G) = \mathcal{N}$ the variety of nilpotent elements in $\mathfrak{g}$.

Lemma 27. *Let $\pi : \hat{G} \to G$ be a central isogeny of reductive groups over k. Then π restricts to a homeomorphism $\pi_{|\hat{\mathcal{U}}} : \hat{\mathcal{U}} \xrightarrow{\sim} \mathcal{U}$, and $d\pi$ restricts to a homeomorphism $d\pi_{|\hat{\mathcal{N}}} : \hat{\mathcal{N}} \xrightarrow{\sim} \mathcal{N}$. If the characteristic of k is 0 or if $d\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ is bijective, these maps are isomorphisms of varieties.*

Proof. Let us recall from [Bor91, §22] that "π is a central isogeny" means that $Z = \ker \pi$ is finite and hence central (we will regard this kernel as a (reduced) group variety rather than as a group scheme), and that $\mathfrak{z} = \ker d\pi$ is central.

First, we note that $\pi_{|\hat{\mathcal{U}}}$ and $d\pi_{|\hat{\mathcal{N}}}$ are bijective; for the latter, this is proved in [J, Prop. 2.6]; the argument for $\pi_{|\hat{\mathcal{U}}}$ is the same.

If $p = 0$, then $d\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism (since the kernel of π is finite). If $d\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism, then $\pi_{|\hat{\mathcal{U}}}$ and $d\pi_{|\hat{\mathcal{N}}}$ are isomorphisms of varieties by [Spr98, Theorem 5.3.2]. Thus, we may now suppose that $p > 0$.

It remains to show that $\pi_{|\hat{\mathcal{U}}}$ and $d\pi_{|\hat{\mathcal{N}}}$ are open maps. Note first that $(*)$ if $f : X \to Y$ is an open map between topological spaces, and $X' \subset X$ satisfies $X' = f^{-1}(f(X'))$ then $f_{|X'} : X' \to f(X')$ is open.

Any surjective morphism of algebraic groups is open; this follows from [Spr98, Theorem 5.1.6(i)]. In particular, $\pi : \hat{G} \to G$ is open, and $d\pi : \hat{\mathfrak{g}} \to d\pi(\hat{\mathfrak{g}})$ is open.

Let $\mathcal{V} = \pi^{-1}(\mathcal{U})$. Then $(*)$ shows that $\pi_{|\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is an open map. Since Z is a finite subgroup of G, the connected components of the variety $\mathcal{V}$ are precisely the sets $z\hat{\mathcal{U}}$ where $z \in Z$. In particular, $\hat{\mathcal{U}}$ is an open subset of $\mathcal{V}$. This shows that $\pi_{|\hat{\mathcal{U}}}$ is open as desired.

Finally, let $\mathcal{M} = d\pi^{-1}(\mathcal{N})$. Since $\mathcal{N} \subset d\pi(\hat{\mathfrak{g}})$, $(*)$ shows that $d\pi_{|\mathcal{M}} : \mathcal{M} \to \mathcal{N}$ is an open map. We claim that the map

$$\Phi : \mathfrak{z} \times \hat{\mathcal{N}} \to \mathcal{M} \quad \text{via } (Z, X) \mapsto Z + X$$

is a homeomorphism. This map is a morphism hence continuous. We will produce an explicit inverse. Let $n \geq 1$ have the property that $X^{[p^n]} = 0$ for all $X \in \hat{\mathcal{N}}$ [it suffices to choose n so that $X^{[p^n]} = 0$ for a *regular* nilpotent element X]. For any $(Z, X) \in \mathfrak{z} \times \hat{\mathcal{N}}$, we have $[Z, X] = 0$ so that $(Z + X)^{[p^n]} = Z^{[p^n]} \in \mathfrak{z}$. Denoting by $\Theta_n : \mathfrak{z} \to \mathfrak{z}$ the "p^n-th root" map as in the remarks following the previous lemma, we may define

$$\Psi : \mathcal{M} \to \mathfrak{z} \times \hat{\mathcal{N}} \quad \text{via } A \mapsto (\Theta_n(A^{[p^n]}), A - \Theta_n(A^{[p^n]})).$$

Then Ψ is continuous by construction, and Φ and Ψ are inverse homeomorphisms.

It now follows that the map $\Gamma : \mathfrak{z} \times \hat{\mathcal{N}} \to \mathcal{N}$ given by $(Z, X) \mapsto d\pi(X)$ is open (since $\Gamma = d\pi_{|\mathcal{M}} \circ \Phi$). If $\mathcal{W} \subset \hat{\mathcal{N}}$ is an open set, then $\mathfrak{z} \times \mathcal{W} \subset \mathfrak{z} \times \hat{\mathcal{N}}$ is an open set. Thus $\Gamma(\mathfrak{z} \times \mathcal{W}) = d\pi(\mathcal{W})$ is open. We have showed that $d\pi_{|\hat{\mathcal{N}}}$ is an open map, which completes the proof of the lemma. □

Lemma 28. *Let $\varepsilon : \mathcal{N} \to \mathcal{U}$ be a G-equivariant homeomorphism. Let $P \subset G$ be a parabolic subgroup with Levi decomposition $P = L \cdot V$. Then ε restricts to a P-equivariant homeomorphism $\mathfrak{v} = \mathrm{Lie}(V) \to V$, and to an L-equivariant homeomorphism $\mathcal{N}(L) \to \mathcal{U}(L)$.*

Proof. Suppose first that $P = B$ is a Borel subgroup with unipotent radical U. Then the proof given in [Car93, Proof of Theorem 5.9.6 (2nd paragraph)] shows that ε induces a homeomorphism $\mathfrak{u} = \mathrm{Lie}(U) \to U$, where U is the unipotent radical of B (in *loc. cit.* one is in the situation where ε is an isomorphism of varieties, but the argument depends only on topological properties of ε).

Now suppose that P is a parabolic subgroup containing B, and that $P = L \cdot U_P$ is a Levi decomposition. Let $\mathcal{N}(L)$ and $\mathcal{U}(L)$ denote the nilpotent and unipotent varieties of L (regarded as subvarieties of $\mathcal{N}$ and $\mathcal{U}$). Let U^- denote the unipotent radical of the Borel group opposite to B. Then we have:

$$\mathcal{U}(L) \cap U = \{u \in U \mid \mathrm{Int}(L)u \cap U^- \neq \emptyset\},$$
$$\mathcal{N}(L) \cap \mathfrak{u} = \{X \in \mathfrak{u} \mid \mathrm{Ad}(L)X \cap \mathfrak{u}^- \neq \emptyset\}$$

and

$$U_P = \{u \in U \mid \mathrm{Int}(P)u \subset U\}, \quad \mathrm{Lie}(U_P) = \{X \in \mathfrak{u} \mid \mathrm{Ad}(P)X \subset \mathfrak{u}\}.$$

The required properties of ε are now immediate from equivariance. □

Proposition 29. *Suppose that p is good for G. Then there is a G-equivariant homeomorphism $\varepsilon : \mathcal{N} \to \mathcal{U}$.*

Proof. Suppose first that G is simply connected and semisimple. Then the result is due to Springer; see [Hu95, Theorem 6.20]. In fact, one gets in this case an isomorphism of varieties; see [BR85, Cor. 9.3.4].

One now deduces the result when G is the product of a torus and a simply connected semisimple group. Since there is a central isogeny from such a group onto our reductive group G [Spr98, Theorem 9.6.5], the result follows from Lemma 27. □

Remark 30. Let $k_0 \subset k$ be a field extension with both k_0 and k algebraically closed. Then the homeomorphism $\varepsilon : \mathcal{N}_{/k} \to \mathcal{U}_{/k}$ of the proposition may be chosen so that its restriction to k_0 points defines a homeomorphism $\mathcal{N}_{/k_0} \to \mathcal{U}_{/k_0}$. Indeed, in the case where G is simply connected, there is an equivariant isomorphism between the two varieties which is defined over $\mathbf{Z}[1/f]$, where f is the product of the bad primes, and hence over k_0; see [Hu95, §6.21]. Thus the claim is true in the simply connected case. For the general statement, there is a central k_0-isogeny from a simply connected group to G. The homeomorphisms in Lemma 27 are then defined over k_0, whence the result in general.

7. Appendix: Springer's isomorphism and p-th powers

Let k and G be as in the previous appendix, and assume that $p > 0$. Denote by $\mathcal{U}$ the variety of unipotent elements in G, and let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}$. We suppose the following hypothesis to hold:

($*$) the group G has a faithful rational representation (ρ, V) for which the trace form $\beta(X,Y) = tr(d\rho(X) \circ d\rho(Y))$ on $\mathfrak{g}$ is nondegenerate.

For convenience, we identify G with a subgroup of $\mathrm{GL}(V)$, and hence also $\mathfrak{g}$ with a subalgebra of $\mathfrak{gl}(V)$. Thus the p-power map $X \mapsto X^{[p]}$ on the p-Lie algebra $\mathfrak{g}$ is the restriction of the usual p-power map in the associative algebra $\mathrm{End}(V)$. The hypothesis ($*$) implies that $\mathfrak{gl}(V) = \mathfrak{m} \oplus \mathfrak{g}$, where $\mathfrak{m} = \mathfrak{g}^{\perp} = \{X \in \mathfrak{gl}(V) \mid \beta(X,Y) = 0 \text{ for each } Y \in \mathfrak{g}\}$.

Let $T \subset G$ be a maximal torus, and let T_1 be any maximal torus of $\mathrm{GL}(V)$ containing T. Write $\mathfrak{h}$ for the Lie algebra of T, and $\mathfrak{h}_1$ for that of T_1.

Lemma 31. $\mathfrak{h}_1 = (\mathfrak{h}_1 \cap \mathfrak{m}) \oplus \mathfrak{h}$. *In particular,* $\mathfrak{h}_1 \cap \mathfrak{m}$ *is the orthogonal complement of* $\mathfrak{h}$ *in* $\mathfrak{h}_1$ *with respect to* β.

Proof. Since the restriction of β to $\mathfrak{h}$ is non-degenerate, the second assertion is a consequence of the first. Let $Y \in \mathfrak{h}_1$, and write $Y = A + B$ with $A \in \mathfrak{g}$ and $B \in \mathfrak{m}$. The first assertion follows if we show that $A \in \mathfrak{h}$. For each weight $\lambda \in X^*(T)$, we have a T-module decomposition $\mathfrak{gl}(V)_\lambda = \mathfrak{g}_\lambda \oplus \mathfrak{m}_\lambda$. This applies in particular for $\lambda = 0$; since $\mathfrak{h} = \mathfrak{g}_0$ we get $\mathfrak{gl}(V)_0 = \mathfrak{h} \oplus \mathfrak{m}_0$. As T acts trivially on $\mathfrak{h}_1$, we have $Y \in \mathfrak{gl}(V)_0$ so that $A \in \mathfrak{h}$ as desired. □

Lemma 32. *For each* $Y \in \mathfrak{h}$, *there is an element* $Z \in \mathfrak{h}$ *with* $Z^{[p]} = Y$.

Proof. It suffices to show that $\mathfrak{h}$ has a k-basis $\{H_i\}$ consisting of elements with $H_i^{[p]} = H_i$; indeed, if there is such a basis, and if $Y = \sum_i a_i H_i$, then $Z = \sum_i a_i^{1/p} H_i$ works (since $\mathfrak{h}$ is Abelian). To see that $\mathfrak{h}$ has such a basis, choose an isomorphism $T \simeq (\mathbf{G}_m)^n$ and use the fact that $\mathrm{Lie}(\mathbf{G}_m)$ has for basis element the $\mathbf{G}_m$-invariant derivation $H = T^{-1}\frac{d}{dT}$ of $k[\mathbf{G}_m] = k[T^{\pm 1}]$, which satisfies $H^{[p]} = H$. □

Lemma 33. *The subspace* $\mathfrak{m}\cap\mathfrak{h}'$ *is invariant by the p-power map: i.e. if* $Y \in \mathfrak{m}\cap\mathfrak{h}'$*, then* $Y^{[p]} \in \mathfrak{m}\cap\mathfrak{h}'$*.*

Proof. We know that $\mathfrak{m}\cap\mathfrak{h}' = \{Y \in \mathfrak{h}' \mid \beta(Y,H) = 0 \text{ for each } H \in \mathfrak{h}\}$. Let $Y \in \mathfrak{m}\cap\mathfrak{h}'$. The lemma then follows if we show that $\beta(Y^{[p]}, H) = 0$ for each $H \in \mathfrak{h}$. Using the previous lemma, we may find an element $Z \in \mathfrak{h}$ with $Z^{[p]} = H$. Then we have $\beta(Y^{[p]}, H) = \beta(Y^{[p]}, Z^{[p]}) = \beta(Y,Z)^p = 0$, where the second equality holds since β is the *trace form* of the representation (ρ, V). This shows that $\beta(Y^{[p]}, H) = 0$, as desired. ☐

Recall that we regard G as a subgroup of $\mathrm{GL}(V)$, and hence as a subset of $\mathrm{End}(V) = \mathfrak{gl}(V)$.

Lemma 34. *Let* $\pi : G \to \mathfrak{g}$ *be the restriction of the orthogonal projection with respect to* β*. Then* $\pi(g^p) = \pi(g)^{[p]}$ *for each* $g \in G$*.*

Proof. Since $g \mapsto \pi(g^p)$ and $g \mapsto \pi(g)^{[p]}$ are morphisms of algebraic varieties, it is enough to show that they coincide on a dense subset of G. The semisimple elements in G constitute such a dense set, by [Hu95, Theorem 2.5]. If $s \in G$ is semisimple, then s lies in a maximal torus T of G by [Spr98, Theorem 6.3.5(i)]. Choose a maximal torus $T_1 \leq \mathrm{GL}(V)$ containing T, and write $\mathfrak{h}$, $\mathfrak{h}_1$ for their Lie algebras as before. Identifying the Lie algebra of $\mathrm{GL}(V)$ with $\mathrm{End}(V)$, we may regard the torus T as a subset of $\mathfrak{h}_1$. In particular, we may regard s as an element of $\mathfrak{h}_1$. According to Lemma 31 we may write $s = A + H$ with $A \in \mathfrak{m}\cap\mathfrak{h}_1$ and $H \in \mathfrak{h}$. Since $\mathfrak{h}_1$ is an Abelian Lie algebra, we have $s^p = s^{[p]} = A^{[p]} + H^{[p]}$. According to Lemma 33 the subspace $\mathfrak{m}\cap\mathfrak{h}_1$ is closed under p-powers; thus $A^{[p]} \in \mathfrak{m}$. It follows that $\pi(s^p) = H^{[p]} = \pi(s)^{[p]}$ as desired. ☐

Theorem 35. *Suppose that G is quasisimple, that p is good for G, and that either* $G = GL_n$ *or G is almost simple and its root system is not of type* A_n*. Then there is a G-equivariant isomorphism of varieties* $L : \mathcal{U} \to \mathcal{N}$ *such that* $L(x^p) = L(x)^{[p]}$ *for all* $x \in \mathcal{U}$*.*

Proof. The existence of a G-morphism L (without the condition on p-th powers) follows from [BR85]. The construction in *loc. cit.* proceeds as follows. If $G = \mathrm{GL}_n$, one takes for L the map $x \mapsto x - 1$ (and the condition on p-powers is clear in that case). Otherwise, according to [SS70, Ch. 1 Lemma 5.3], a group isogenous to G satisfies the hypothesis $(*)$ and such that the identity endomorphism of the representation V is orthogonal via the trace form β to every element of $\mathfrak{g}$ (equivalently: each element of $\mathfrak{g}$ has trace 0). According to the summary in [Hu95, §0.13], $\mathfrak{g}$ is a simple Lie algebra under our assumptions. If $\tilde{G}$ is a group isogenous to G, the isogeny thus induces an isomorphism on Lie algebras. Since the isomorphisms of Lemma 27 evidently respect the p-power operations, we may if necessary replace G with an isogenous group and so suppose that G itself satisfies (*).

One then considers the map $\pi : G \to \mathfrak{g}$ which is the restriction of the orthogonal projection with respect to the trace form. Then π is clearly G-equivariant, and satisfies $\pi(1) = 0$. According to [BR85], $L = \pi_{|\mathcal{U}} : \mathcal{U} \to \mathcal{N}$ is a G-isomorphism of varieties. It follows from the previous lemma that $L(x^p) = L(x)^{[p]}$, whence the theorem. ☐

Remark 36. This theorem permits a simplification of the proof of Testerman's "order formula" given in [M02]. It allows one to deduce the order formula for unipotent elements from the corresponding formula for the p-nilpotence degree of nilpotent elements; the comparison of these respective "orders" was achieved in a different way in *loc. cit.*

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