REDUCTIVE SUBGROUP SCHEMES OF A PARAHORIC GROUP SCHEME

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ABSTRACT. Let K be the field of fractions of a complete discrete valuation ring $\mathcal{A}$ with residue field k, and let G be a connected reductive algebraic group over K. Suppose $\mathcal{P}$ is a parahoric group scheme attached to G. In particular, $\mathcal{P}$ is a smooth affine $\mathcal{A}$-group scheme having generic fiber $\mathcal{P}_{\mathrm{K}} = G$; the group scheme $\mathcal{P}$ is in general not reductive over $\mathcal{A}$.

If G splits over an unramified extension of K, we find in this paper a closed and reductive $\mathcal{A}$-subgroup scheme $\mathscr{M} \subset \mathcal{P}$ for which the special fiber $\mathscr{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$. Moreover, we show that the generic fiber $M = \mathscr{M}_{\mathrm{K}}$ is a subgroup of G which is geometrically of type $C(\mu)$ – i.e. after a separable field extension, M is the identity component $M = C_G^o(\phi)$ of the centralizer of the image of a homomorphism $\phi : \mu_n \to H$, where μ_n is the group scheme of *n-th roots of unity* for some $n \geq 2$. For a connected and split reductive group H over any field $\mathcal{F}$, the paper describes those subgroups of H which are of type $C(\mu)$.

CONTENTS

1. INTRODUCTION

1.1. **The connected centralizer of a μ-homomorphism.** Consider a field $\mathcal{F}$ and a connected and reductive algebraic group G over $\mathcal{F}$. In sections 2 and 3 of this paper, we are going to describe a certain class of connected and reductive subgroups of G which we call the subgroups of type $C(\mu)$.

For $n \geq 2$, denote by μ_n the group scheme of *n-th roots of unity*; over the field $\mathcal{F}$, μ_n is represented by the $\mathcal{F}$-algebra $\mathcal{F}[T]/\langle T^n - 1\rangle$. If $\mathcal{F}$ has characteristic 0, μ_n is always a smooth group scheme, but μ_n is not smooth if $\mathcal{F}$ has characteristic $p > 0$ and $n \equiv 0 \pmod{p}$.

We say that a connected subgroup M of G is of type $C(\mu)$ if M is the identity component of the centralizer in G of the image of a homomorphism $\phi : \mu_n \to G$ [1]. Moreover, M is *geometrically* of type $C(\mu)$ if M_{L} is of type $C(\mu)$ for some finite, separable field extension $\mathrm{K} \subset \mathrm{L}$; the example of Proposition 3.6.3 exhibits a group which is geometrically of type $C(\mu)$ but not itself of type $C(\mu$.

If μ_n is smooth, then M is the centralizer of some (semisimple) element $\zeta \in G(\mathrm{L})$ in the image of ϕ for some finite separable field extension L of K, but for example if $n = p$ where $p > 0$ is the characteristic of K, the group $\mu_p(\mathcal{F}_{\mathrm{alg}})$ of points over an algebraic closure of $\mathcal{F}$ is trivial, and instead M is the centralizer of a semisimple element $X \in \mathrm{Lie}(G)$.

In section 3 of this paper, we examine such connected centralizers. Among other things, we show that the image of ϕ lies in a maximal torus T of G; see Proposition 3.4.1. In particular, this allows us to deduce that a subgroup of type $C(\mu)$ is reductive and contains a maximal torus of G; if M (and thus

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[1] We will permit ourselves to write $C_G(\phi)$ for the centralizer of the image of ϕ, and $C_G^o(\phi)$ for the identity component of this centralizer.

also G) are moreover split, it follows that M is determined by a subsystem Ψ of the root system Φ of G (relative to a split maximal torus T).

The manuscript **[mcninch03:MR1976698]** used the terminology *pseudo-Levi subgroups* of G to describe the subgroups of the form $L = C^o_G(s)$ for semisimple elements $s \in G$ [2]. Any pseudo-Levi subgroup is of type $C(\mu)$, and if $\mathcal{F}$ has characteristic zero, a subgroup M is pseudo-Levi if and only if it is of type $C(\mu)$. But in general there are subgroups of type $C(\mu)$ which are not pseudo-Levi subgroups in the sense of this older paper; see e.g. the examples in Section 3.6.

Now suppose that G is split and that T is a maximal k-split torus. We introduce an equivalence relation on homomorphisms $\phi : \mu_n \to T$ for varying n, and we call the classes "μ-homomorphisms". Then the group of μ-homomorphisms with values in T may be identified with $Y \otimes \mathbf{Q}/\mathbf{Z}$ where $Y = X_*(T)$ is the group of cocharacters of T. Let $x \in Y \otimes \mathbf{Q}$, write $\overline{x}$ for its class in $Y \otimes \mathbf{Q}/\mathbf{Z}$, let $\phi_{\overline{x}}$ denote the equivalence class of a homomorphism $\psi : \mu_n \to T$, and let $M = C^o_G(\phi_{\overline{x}}) = C^o_G(\psi)$ the corresponding subgroup of G of type $C(\mu)$.

In section 2, we describe a subsystem Φ_x of the root system Φ determined in a natural way by the point $x \in Y \otimes \mathbf{Q}$ – see section 2.5. In fact, Φ_x can be characterized as the root system whose Dynkin diagram is obtained by "removing certain nodes (determined by x) from the extended Dynkin diagram of G" – see Remark 2.5.4. We show that $M = C^o_G(\phi_{\overline{x}})$ has root system Φ_x; see Theorem 3.4.6.

1.2. **Levi decomposition of the special fiber of a parahoric group scheme.** Let H be a linear algebraic group over any field $\mathcal{F}$. We say that the unipotent radical of H is defined over $\mathcal{F}$ if there is an $\mathcal{F}$-subgroup $R \subset H$ such that $R_{\overline{\mathcal{F}}}$ coincides with the *geometric* unipotent radical $R_u(H_{\overline{\mathcal{F}}})$; if moreover R is an $\mathcal{F}$*-split* unipotent group, we say that the unipotent radical of H is defined and split over $\mathcal{F}$. If $\mathcal{F}$ is imperfect, there are examples of groups H whose unipotent radical is not defined over $\mathcal{F}$.

If the unipotent radical of H is defined over $\mathcal{F}$, write $\pi : H \to H/R_uH$ for the quotient mapping. To say that H has a *Levi decomposition* (over $\mathcal{F}$) means that there is a (necessarily reductive) $\mathcal{F}$-subgroup M of H such that the restriction $\pi_{|M} : M \to H/R_uH$ is an isomorphism [3]

Our interest here is in linear algebraic groups which arise when considering reductive groups over *local field*. Thus, suppose that K is a local field, by which we mean the field of fractions of a complete discrete valuation ring $\mathcal{A}$. Write k for the residue field of $\mathcal{A}$. We make no assumptions on k - in particular, we do not require k to be perfect.

Consider a extension field $L \supset K$ of finite degree, and let $\mathcal{B}$ denote the integral closure of $\mathcal{A}$ in L; then $\mathcal{B}$ is also a discrete valuation ring, say with residue field ℓ. Recall that L is *unramified* over K provided that ℓ is a *separable* extension of k with $[L : K] = [\ell : k]$.

We now consider a reductive algebraic group G over the field K. Following the work of F. Bruhat and J. Tits, one can view G as the generic fiber $G = \mathcal{P}_K$ of various smooth affine group schemes $\mathcal{P}$ over $\mathcal{A}$ which we will refer to as *parahoric group schemes*; see Definitions 4.5.3 and 4.3.4. In this manuscript, we only consider parahoric group schemes $\mathcal{P}$ for which the generic fiber $G = \mathcal{P}_K$ splits over an unramified extension of K; under this assumption, the fibers of $\mathcal{P}$ are *connected* – see Theorem 4.2.2.

In fact, the group of K-rational points $G(K)$ of G acts on the Bruhat-Tits building of G, and the subgroup of $\mathcal{A}$-rational points $\mathcal{P}(\mathcal{A}) \subset G(K)$ – known as a parahoric subgroup – is very closely related to the stabilizer of a point for this action. If the residue field k is finite, the $\mathcal{P}(\mathcal{A})$ are compact open subgroups of the locally compact group $G(K)$.

The special fiber $\mathcal{P}_k$ is a connected linear algebraic group over k which in general is not reductive. Under the additional assumptions that the residue field k is perfect and that G splits over an unramified extension of k, we proved in **[mcninch10:MR2753264]** that (*) the identity component $\mathcal{P}_k$ of the special fiber has a *Levi decomposition*.

In the present work, we improve the result (*) with no assumption on the residue field k, under the assumption that G splits over an unramified extension. We show under these assumptions that that

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[2]This class of subgroups was viewed as a generalization of the class of all Levi factors of parabolic subgroups of G

[3]If the unipotent radical of H fails to be defined over $\mathcal{F}$, H may still possess a *Levi factor* M – i.e. an $\mathcal{F}$-subgroup M for which $\pi : M_{\overline{\mathcal{F}}} \to H_{\overline{\mathcal{F}}}/R_u(H_{\overline{\mathcal{F}}})$ is an isomorphism where $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$. In this paper, we only consider Levi decompositions of groups whose unipotent radicals are defined over the ground field.

the unipotent radical of $\mathcal{P}_{\mathrm{k}}$ is defined and split over k – see Proposition 4.3.7(a) and Proposition 4.5.5 –, and that a Levi factor of $\mathcal{P}_{\mathrm{k}}$ can be realized as the special fiber of a reductive subgroup scheme of $\mathcal{P}$. Let T be a maximal K-split torus, write $\mathscr{T}$ for "the" $\mathcal{A}$-split torus with generic fiber T, and let $\mathcal{P}$ be a parahoric group scheme containing $\mathscr{T}$. In the final sections of this paper – see Section 4.4 and Section 4.5 – we are going to prove:

Theorem 1. *Assume that G splits over an unramified extension of* K. *There is a reductive $\mathcal{A}$-subgroup scheme $\mathscr{M} \subset \mathcal{P}$ containing $\mathscr{T}$ such that*

(a) The special fiber $\mathcal{P}_{\mathrm{k}}$ has a Levi decomposition with Levi factor $\mathscr{M}_{\mathrm{k}}$, and
(b) The generic fiber $\mathscr{M}_{\mathrm{K}}$ is a connected reductive subgroup of G of type $C(\mu)$ which contains T.

When G is split over K, a parahoric subgroup $\mathcal{P} = \mathcal{P}_x$ containing $\mathscr{T}$ is determined by the choice of an element $x \in V = Y \otimes \mathbf{Q}$. If $\overline{x}$ is the class of x in $Y \otimes \mathbf{Q}/\mathbf{Z}$, the reductive subgroup scheme $\mathscr{M}$ in the Theorem is precisely the connected centralizer $C^o_{\mathcal{P}}(\phi_{\overline{x}})$ where $\phi_{\overline{x}} : \mu \to \mathscr{T}$ is the μ-homomorphism determined by $\overline{x}$; see Theorem 4.4.2.

We remark that Theorem 1 plays an important role in our recent manuscript **[mcninch16:nilpotent-orbits-over-local-fie** which relates nilpotent orbits on the reductive quotient of the special fiber of a parahoric group scheme $\mathcal{P}$ with the nilpotent orbits on the generic fiber $\mathcal{P}_{\mathrm{K}}$.

Finally, we remark that after completion of this paper, an anonymous referee pointed out to us that in the manuscript [**tits90:MR1058572**], Jacque Tits considers split reductive groups G over the field $\mathrm{k}((t))$; when $\mathcal{P}$ is a maximal parahoric, Tits finds in that paper the group scheme $\mathscr{M}$ of Theorem 1 and shows that $\mathscr{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$.

1.3. **Notation and terminology.** By a *linear algebraic group H* over a field $\mathcal{F}$, we mean an affine group scheme which is smooth and of finite type over $\mathcal{F}$; this amounts to a the same thing as a (reduced) linear algebraic group defined over $\mathcal{F}$ as in [**borel91:MR1102012**] or [**springer98:MR2458469**]. Unless otherwise indicated, by a subgroup of H we mean a closed $\mathcal{F}$-subgroup scheme.

Suppose that $\mathcal{A}$ is a local integral domain with fractions K and residue field k. If X is a separated scheme of finite type over $\mathcal{A}$, the *generic fiber* of X is the K-variety $X_{\mathrm{K}} = X \times_{\mathrm{Spec}(\mathcal{A})} \mathrm{Spec}(\mathrm{K})$ obtained by base change, and the *special fiber* of X is the k-variety $X_{\mathrm{k}} = X \times_{\mathcal{A}} \mathrm{K} = X \times_{\mathrm{Spec}(\mathcal{A})} \mathrm{Spec}(\mathrm{k})$ obtained by base change.

If X is a group scheme over $\mathcal{A}$, then X_{K} is a K-group scheme and X_{k} is a k-group scheme. If X is smooth, affine, and of finite type over $\mathcal{A}$, then X_{K} and X_{k} are *linear algebraic groups*.

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2. Certain sub-systems of a root system

The goal of this section is to describe certain subsystems Ψ of a root system Φ. Probably most of the material in this section is well-known, though I don't really know of a concise account for some of the results. Some of this material appears e.g. in the papers [**reeder10:MR2674853**] and [**reeder12:MR3000483**].

The subsystems Ψ depend on the choice of a point x in the linear space affording the *reflection representation* of the Weyl group of Ψ. They are described in section 2.5 – see especially Theorem 2.5.3. More precisely, Ψ depends only on the facet F containing x, where the facets are defined with respect to the affine root system. As perhaps a justification for the level of detail given here, note that our exposition confirms that the subsystems Ψ of interest can all be obtained using points x taken from a $\mathbf{Q}$-form of the reflection representation; see especially Proposition 2.4.2.

Suppose that G is a split reductive group over a field with root system Φ. In the terminology used in the introduction, the subsystems we describe will be precisely the root systems of the split reductive subgroups M of G which are of type $C(\mu)$; these subgroups will be studied in section 3.

Throughout this section, we will adopt the following notations: V will denote a finite dimensional vector space over the field $\mathbf{Q}$ of rational numbers, and q will denote a positive definite quadratic form q on V. We write $\langle v, w\rangle \in \mathbf{Q}$ for the value of the associated bilinear form at $v, w \in V$; thus $q(v) = \langle v, v\rangle$. The bilinear form provides an identification of V with its dual space $V^\vee$; in what follows, we freely apply this identification when making reference to results from **[bourbaki02:MR1890629]**.

Given any field extension $\mathbf{Q} \subset \mathrm{E}$, we form the tensor product $V_{\mathrm{E}} = V \otimes_{\mathbf{Q}} \mathrm{E}$. The quadratic form q determines by extension of scalars a non-degenerate form on V_{E} which we also denote by q. If E has a real embedding, then q remains positive definite on V_{E}. In particular, the form q determines the Euclidean metric topology on $V_{\mathbf{R}}$.

2.1. **Simplicial cones and simplices.** Let $w_0 \in V$ and let $v_1, \dots, v_d \in V$ be $\mathbf{Q}$-linearly independent vectors in V. For a field E with an embedding in $\mathbf{R}$, the *rational simplicial* E*-cone* based at w_0 determined by the vectors $\vec{v}$ is the subset

$$C_F = C_F(w_0, v_1, \dots, v_d) = \left\{ w_0 + \sum_{j=1}^{d} a_j v_j \mid a_j \in \mathrm{E}_{>0} \quad \forall j = 1, \dots, d \right\}$$

Lemma 2.1.1. *For w_0 and $v_1, \dots, v_d$ as above, $C_{\mathbf{Q}} = C_{\mathbf{R}} \cap V$, and $C_{\mathbf{Q}}$ is dense in $C_{\mathbf{R}}$ for the Euclidean topology on $V_{\mathbf{R}}$.*

Now fix points $w_0, w_1, \dots, w_d \in V$ which are *affinely independent*. A convex combination $\sum_{i=0}^{d} t_i w_i$ of these points is determined by positive scalars $t_0, t_1, \dots, t_d$ for which $\sum_{i=0}^{d} t_i = 1$.

If E is a field with a real embedding, the E-*simplex* determined by the points $w_0, \dots, w_d$ is the set

$$S_{\mathrm{E}} = S_{\mathrm{E}}(w_0, \dots, w_d) = \left\{ \sum_{i=0}^{d} t_i w_i \mid t_i \in \mathrm{E}_{>0}, \quad \sum_{i=0}^{d} t_i = 1 \right\}.$$

Lemma 2.1.2. *Let $w_0, \dots, w_d$ affinely independent. Then $S_{\mathbf{Q}} = S_{\mathbf{R}} \cap V$ and $S_{\mathbf{Q}}$ is dense in $S_{\mathbf{R}}$ for the Euclidean topology on $V_{\mathbf{R}}$.*

Now, the results Lemmas 2.1.1 and 2.1.2 are both immediate consequences of the fact that $\mathbf{Q}$ is dense in $\mathbf{R}$ for the Euclidean topology.

2.2. **Root systems and the finite Weyl group.** Let $\Phi \subset V$ be a (reduced) root system in the span $V_e = \mathbf{Q}\Phi \subset V$; see **[bourbaki02:MR1890629]** for definitions. For $\alpha \in \Phi$, write $\alpha^\vee = 2\alpha/q(\alpha) \in V_e$. Recall **[bourbaki02:MR1890629]** that:

(i) the reflection $s_\alpha \in O(V, q)$ given by the rule $s_\alpha(v) = v - \langle v, \alpha^\vee\rangle\alpha$ is the orthogonal reflection s_H in the hyperplane $H = H_\alpha = \alpha^\perp$, and
(ii) $\langle \Phi, \alpha^\vee\rangle \subset \mathbf{Z}$.

Write

$$\mathcal{H} = \{H_\alpha = \alpha^\perp \mid \alpha \in \Phi\}$$

for the system of linear hyperplanes in V determined by Φ. The *Weyl group* of Φ is the finite reflection group $W = W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi\rangle = \langle s_H \mid H \in \mathcal{H}\rangle \subset O(V, q)$.

Proposition 2.2.1. *[**bourbaki02:MR1890629**] Write $V = \displaystyle\bigoplus_{i=0}^{d} V_i$ where $V_0 = V^W$ and $V_1, \dots, V_d$ are non-trivial irreducible $\mathbf{Q}W$-representations. Then $V_e = \bigoplus_{i=1}^{d} V_i$, and if $1 \le i \le d$, then:*

(a) V_i is an absolutely irreducible $\mathbf{Q}W$-module.
(b) $\Phi_i = \Phi \cap V_i$ is a root system in V_i.

(c) If $W(\Phi_i)$ denotes the Weyl group of Φ_i, and if W_i is the subgroup of W acting trivially on $V_i^{\perp}$, then W_i identifies with $W(\Phi_i)$ and W is the direct product $W = W_1 \times \cdots \times W_p$.

The root systems $\Phi_i \subset \Phi$ of the preceding Proposition are the *irreducible components* of the root system Φ.

Remark 2.2.2. Let $\mathcal{F}$ be a field, and let G be a split reductive group over $\mathcal{F}$. The choice of a maximal split torus T, determines the *root datum* $(X, Y, \Phi, \Phi^\vee)$ of G relative to T. Here, $X = X^*(T)$ is the character group of T, $Y = X_*(T)$ is the cocharacter group of T, $\Phi \subset X$ is the set of roots, and $\Phi^\vee \subset Y$ is the set of co-roots. In this situation, the Weyl group acts on both X and on Y. We take $V = Y \otimes_{\mathbf{Z}} \mathbf{Q}$ and fix a W-invariant positive definite bilinear form on V. Then X may be identified with the lattice $\{x \in V \mid \langle x, Y \rangle \subset \mathbf{Z}\}$. In particular, with these identifications, the roots Φ are contained in V, and in this way the results described in this section can be applied to the root system of the group G.

2.3. Affine hyperplanes and affine Weyl group.

Consider the group $\mathrm{Aff}(V)$ of all *affine displacements* of V; it is the semidirect product of $\mathrm{GL}(V)$ and the normal subgroup of "translations by V":

$$\mathrm{Aff}(V) \simeq V \rtimes \mathrm{GL}(V).$$

A root $\alpha \in \Phi$ and an integer ℓ together determine an affine function

$$a(\alpha, \ell) : V \to \mathbf{Q} \quad \text{by the rule} \quad a(\alpha, \ell)(w) = \langle \alpha, w \rangle - \ell,$$

and an affine hyperplane

$$H_{\alpha,\ell} = a(\alpha, \ell)^{-1}(0) = \{w \in V \mid \langle \alpha, w \rangle = \ell\} \subset V.$$

Write

$$\widetilde{\mathcal{H}} = \{H_{\alpha,\ell} \mid \alpha \in \Phi, \ell \in \mathbf{Z}\}$$

for the collection of these hyperplanes in V.

For $H \in \widetilde{\mathcal{H}}$, write $s_H \in \mathrm{Aff}(V)$ for the orthogonal (affine) reflection in the hyperplane H. The affine Weyl group is the subgroup

$$W_{\mathrm{aff}} = \langle s_H \mid H \in \widetilde{\mathcal{H}} \rangle \subset \mathrm{Aff}(V).$$

The action by conjugation of W_{aff} on its normal subgroup V determines a linear representation $U : W_{\mathrm{aff}} \to \mathrm{GL}(V)$.

Proposition 2.3.1. *[**bourbaki02:MR1890629**] The image of U is the Weyl group W of the root system Φ, the kernel of U is the subgroup $\mathbf{Z}\Phi^\vee$ of the translation subgroup $V \subset \mathrm{Aff}(V)$, and W_{aff} is the semidirect product of W and the subgroup $\mathbf{Z}\Phi^\vee$.*

Note that $\mathcal{H} \subset \widetilde{\mathcal{H}}$. For $\alpha \in \Phi$ and $\ell \in \mathbf{Z}$, the affine function $a(\alpha, \ell)$ has a unique extension to an affine function on $V_{\mathbf{R}}$, and its zero-locus is a hyperplane $H_{\alpha,\ell,\mathbf{R}}$ in $V_{\mathbf{R}}$. We thus find systems of hyperplanes $\mathcal{H}_{\mathbf{R}}$ and $\widetilde{\mathcal{H}}_{\mathbf{R}}$ in $V_{\mathbf{R}}$ determined by Φ, and bijective mappings $\mathcal{H} \to \mathcal{H}_{\mathbf{R}}$ and $\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}_{\mathbf{R}}$ given by $H \mapsto H_{\mathbf{R}}$.

2.4. Facets, chambers and alcoves.

As in [**bourbaki02:MR1890629**], we may speak of the *facets* $\mathbf{F}$ in $V_{\mathbf{R}}$ for the system of affine hyperplanes $\widetilde{\mathcal{H}}_{\mathbf{R}} = \{H_{\alpha,\ell,\mathbf{R}}\}$ in $V_{\mathbf{R}}$, and for the system of linear hyperplanes $\mathcal{H}_{\mathbf{R}} = \{H_{\alpha,\mathbf{R}}\}$ in $V_{\mathbf{R}}$.

By a *facet* F in V relative to $\widetilde{\mathcal{H}}$ resp. relative to $\mathcal{H}$, we shall mean the intersection $F = \mathbf{F} \cap V$ of V with a facet $\mathbf{F}$ in $V_{\mathbf{R}}$ relative to $\widetilde{\mathcal{H}}_{\mathbf{R}}$ resp. relative to $\mathcal{H}_{\mathbf{R}}$. If $F = \mathbf{F} \cap V$, the *closure* of F is by definition the set $\overline{F} = \overline{\mathbf{F}} \cap V$ where $\overline{\mathbf{F}}$ is the closure of $\mathbf{F}$ for the Euclidean topology on $V_{\mathbf{R}}$.

A *chamber* of V is a facet C relative to $\mathcal{H}$ which is contained in no hyperplane $H \in \mathcal{H}$. An *alcove* of V is a facet A relative to $\widetilde{\mathcal{H}}$ which is contained in no hyperplane $H \in \widetilde{\mathcal{H}}$.

Proposition 2.4.1. *Let $F = V \cap \mathbf{F}$ be a facet in V relative to $\mathcal{H}$ resp. $\widetilde{\mathcal{H}}$, where $\mathbf{F}$ is a facet in $V_{\mathbf{R}}$ relative to $\mathcal{H}_{\mathbf{R}}$ resp. $\widetilde{\mathcal{H}}_{\mathbf{R}}$. Then:*

(a) F is dense in $\mathbf{F}$ for the Euclidean topology on $V_{\mathbf{R}}$, and

(b) If F' is another facet in V relative to $\mathcal{H}$ resp. $\widetilde{\mathcal{H}}$, then $F = F'$ if and only if $\mathbf{F} = \mathbf{F}'$.

Proof. Of course, (b) is a consequence of (a).

To prove (a), first recall the decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$ of Proposition 2.2.1. Write Φ as the union of its irreducible subsystems $\Phi_i \subset V_i$. For $1 \le i \le d$, consider the systems of hyperplanes $\mathcal{H}_i = \{H_\alpha \mid \alpha \in \Phi_i\}$ and $\widetilde{\mathcal{H}}_i = \{H_{\alpha,\ell} \mid \alpha \in \Phi_i, \ell \in \mathbf{Z}\}$ in V; each H in either $\widetilde{\mathcal{H}}_i$ or $\mathcal{H}_i$ determines a hyperplane $H \cap V_i$ in V_i.

If follows from **[bourbaki02:MR1890629]** that $F_\mathbf{R} = V_{0,\mathbf{R}} \times \mathbf{F}_1 \times \cdots \times \mathbf{F}_d$ where $\mathbf{F}_i$ is a facet in $V_{i,\mathbf{R}}$ for the system of hyperplanes determined by $\mathcal{H}_{i,\mathbf{R}}$ resp. $\widetilde{\mathcal{H}}_{i,\mathbf{R}}$.

If F is a facet relative to $\widetilde{\mathcal{H}}$, then $\mathbf{F}_i$ is a facet relative to $\widetilde{\mathcal{H}}_{i,\mathbf{R}}$, and according to **[bourbaki02:MR1890629]**, there is an affinely independent collection of points $w_0, \dots, w_d \in V$ such that $\mathbf{F}_i$ is the real simplex $S_\mathbf{R}(w_0, \dots, w_d)$ determined by these points as in section 2.1. Now Lemma 2.1.2 shows that F_i is the $\mathbf{Q}$-simplex $S_\mathbf{Q}(w_0, \dots, w_d)$ and that F_i is dense in $\mathbf{F}_i$ as required.

If instead F is a facet relative to $\mathcal{H}$, then $\mathbf{F}_i$ is a facet relative to $\mathcal{H}_{i,\mathbf{R}}$ and according to **[bourbaki02:MR1890629]** there is a linearly independent subset $v_1, \dots, v_d$ of V such that $\mathbf{F}_i$ is the real simplicial cone $C_\mathbf{R}(0, v_1, \dots, v_d)$ as in section 2.1. Now, Lemma 2.1.1 shows that F_i is the $\mathbf{Q}$-simplicial cone $C_\mathbf{Q}(0, v_1, \dots, v_d)$ and that F_i is dense in $\mathbf{F}_i$ as required. □

Proposition 2.4.2. *(a) The Weyl group W acts on the set of all facets in V relative to $\mathcal{H}$ and preserves the subset of all chambers. Moreover, the closure of a chamber is a fundamental domain for the action of W on V.*
(b) The affine Weyl group W_{aff} acts on the set of all facets in V relative to $\widetilde{\mathcal{H}}$ and preserves the subset of all alcoves. Moreover, the closure of an alcove is a fundamental domain for the action of W_{aff} on V.

Proof. The analogous statements are known to hold for the actions of W and W_{aff} on the Euclidean space $V_\mathbf{R}$; see **[bourbaki02:MR1890629]**. Since by Proposition 2.4.1 a facet $F = V \cap \mathbf{F}$ is *dense* in $\mathbf{F}$ for the Euclidean topology, and since the action of an element of W or W_{aff} on $V_\mathbf{R}$ is continuous, the conclusion follows at once. □

2.5. **The root subsystem associated to a point in V.** The subset $\Psi \subset \Phi$ is said to be *closed* if whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Psi$. The subset Ψ is *symmetric* if $\Psi = -\Psi = \{-\alpha \mid \alpha \in \Psi\}$. If Ψ is closed and symmetric, it is again a root system.

We now fix a point $x \in V$, and we put

$$\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x \rangle \in \mathbf{Z}\}.$$

Note that Φ_x is evidently a *closed* and *symmetric* subset of Φ; hence Φ_x is a root system.

Verification of the following Proposition is immediate:

Proposition 2.5.1. *(a) Φ_x depends only on the W_{aff} orbit of x; i.e. $\Phi_{wx} = \Phi_x$ for all $w \in W_{\mathrm{aff}}$.*
(b) If $\Phi = \Phi_1 \cup \cdots \cup \Phi_d$ is the decomposition of Φ into irreducible components, then Φ_x is the disjoint union of the subsystems $\Phi_{i,x}$ for $1 \le i \le d$.

In view of this Proposition, in order to describe Φ_x it is sufficient to suppose that Φ is irreducible. Moreover, we may even choose an "optimal" representative for the W_{aff}-orbit of x. The following Lemma provides a useful representative from our point of view.

Let A be an alcove in V relative to $\widetilde{\mathcal{H}}$. A *face* of A is a facet F contained in $\overline{A}$ for which $F \subset H$ for precisely one hyperplane $H \in \widetilde{\mathcal{H}}$; if F is a face of A and $F \subset H \in \widetilde{\mathcal{H}}$, we say that the hyperplane H is a *wall* of A.

Proposition 2.5.2. *Let Φ be an irreducible root system, and recall that $V_e = V_1$ is the $\mathbf{Q}$-span of Φ in V. Let $\alpha_1, \dots, \alpha_\ell$ be a system of simple roots, and let $\varpi_1, \dots, \varpi_\ell \in V_1$ be elements with $\langle \alpha_i, \varpi_j \rangle = \delta_{i,j}$. Write $\alpha_0 = -\widetilde{\alpha}$ where*

$$\widetilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i$$

is the highest root *in Φ for this choice of simple roots, and let $\varpi_0 = 0$.*

(a) The simplex $A = C(\varpi_0, \varpi_1/n_1, \ldots, \varpi_\ell/n_\ell)$ is an alcove for the root system $\widetilde{\mathcal{H}}$.
(b) The walls of A are the hyperplanes defined by the affine functions
$$\{a(\alpha_0, 1)\} \cup \{a(\alpha_i, 0) \mid 1 \leq i \leq \ell.\}$$

Proof. (a) follows from [**bourbaki02:MR1890629**]; (b) then follows from definitions. □

If the affine hyperplane $H \in \widetilde{\mathcal{H}}$ is the zero locus of the affine function $a(\alpha, d)$, we say that H is *labeled by* α.

Theorem 2.5.3. *Suppose that Φ is irreducible, and that $\alpha_1, \ldots, \alpha_\ell$ is a system of simple roots. Let A be the alcove of Proposition 2.5.2; thus A is the simplex determined by 0 and the elements ϖ_i/n_i. Now suppose that $x \in \overline{A}$. Define*
$$J = \{i \in \{0, 1, \ldots, \ell\} \mid x \text{ lies on the wall of } A \text{ labeled by } \alpha_i\}.$$
Then $\{\alpha_j \mid j \in J\}$ is a system of simple roots for Φ_x.

Proof. Since any proper subset of the roots $\{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ is $\mathbf{Q}$-linearly independent in V, we only must argue that any $\alpha \in \Phi_x$ is a $\mathbf{Z}$-linear combination of the indicated roots.

Fix $\alpha \in \Phi_x$, let $n = \langle \alpha, x \rangle \in \mathbf{Z}$, and consider the affine reflection $s_{\alpha,n}$ – i.e. the reflection in the hyperplane $H = \{y \in V \mid \alpha(y) = n\}$. Then $s_{\alpha,n}$ fixes x.

It follows from [**bourbaki02:MR1890629**] that the stabilizer W_x of x in W_{aff} is the subgroup $W_x = W_J = \langle s_{H_j} \rangle$ where H_j is the wall of A labeled by the root α_j.

Since $s_{\alpha,n}$ is a reflection in the Coxeter group W_x, it follows from [**bourbaki02:MR1890629**] that $s_{\alpha,n}$ is W_x-conjugate to s_{H_j} for some $j \in J$; it now follows that α is a $\mathbf{Z}$-linear combination of the roots $\{\alpha_j \mid j \in J\}$, as required. □

Remark 2.5.4. The irreducible root system Φ is determined by its *Dynkin diagram*; see [**bourbaki02:MR1890629**]. With notation as before, the roots $\{\alpha_0, \ldots, \alpha_\ell\}$ label the nodes of the *completed Dynkin diagram loc. cit.* (VI.4.3). Now, the Theorem implies that the (ordinary) Dynkin diagram of the - in general, reducible - root system Φ_x is obtained by discarding those vertices of this completed diagram labeled by the simple roots α_t for $t \notin J$ together with any edges connected to such a discarded vertex, where $J \subset \{0, \ldots, \ell\}$ is the subset found in the statement of Theorem 2.5.3.

3. μ-HOMOMORPHISMS AND THEIR CENTRALIZERS

Fix throughout Section 3 a connected and reductive algebraic group G over the ground field $\mathcal{F}$.

Consider an $\mathcal{F}$-homomorphism $\phi : \mu_n \to G$, where μ_n is the finite group scheme of *n-th roots of unity*. The main goal of this section is to describe the connected centralizer $C_G^o(\phi)$ of the image of ϕ. Recall from the introduction that M is a *subgroup of type* $C(\mu)$ if $M = C_G^o(\phi)$ for some ϕ as above.

We are going to show that the image of ϕ always lies in some maximal torus T of G - see Proposition 3.4.1. As a consequence, we see that M is a reductive subgroup containing T; see Proposition 3.1.2.

We define in section 3.4 an equivalence relation on the collection of homomorphisms $\psi : \mu_n \to T$ for varying n; the equivalence classes are called μ-homomorphisms. If T is a split torus, the collection of μ-homomorphisms with values in T is in bijection with the group $Y \otimes \mathbf{Q}/\mathbf{Z}$, where $Y = X_*(T)$ is the group of cocharacters of T. Suppose that the μ-homomorphism $\phi = \phi_{\overline{x}}$ is determined by the class $\overline{x}$ in $Y \otimes \mathbf{Q}/\mathbf{Z}$ of the point $x \in Y \otimes \mathbf{Q}$. We will show – see Theorem 3.4.6 - that the group $M = C_G^o(\phi)$ of type $C(\mu)$ has root system Φ_x as described in Theorem 2.5.3.

We recall that the so-called Borel-de Siebenthal procedure – first described in [**borel49:MR0032659**] – can be used to obtain the root systems of *all* split reductive subgroups containing a maximal torus; it amounts to recursive application of the procedure described in Remark 2.5.4. The subgroups of type $C(\mu)$ do not exhaust all such subgroups.

Related results and descriptions of centralizers are studied e.g. in [**humphreys95:MR1343976**], in [**lusztig95:MR1369407**] – see especially Lemma 5.4 in that citation –, and in [**mcninch03:MR1976698**]. In [**Oberwolfach-Serre**], J-P. Serre suggested to replace the "semisimple elements of finite order" by μ-homomorphisms. Finally, note that the recent paper **[pepin-lehalleur15:MR3525844]** has some overlap with the point of view taken here.

3.1. **The centralizer of a diagonalizable subgroup scheme.** The affine group scheme M is *diagonalizable* if the Abelian group of its characters $X^*(M) = \operatorname{Hom}_{\mathcal{F}-\mathrm{gp}}(M, \mathbf{G}_m)$ forms a basis of the coordinate algebra $\mathcal{F}[M]$ as a linear space. More generally, M is of *multiplicative type* if there is a finite separable extension $E/\mathcal{F}$ for which the E-group M_E obtained by base change is diagonalizable. For example, a torus T is of multiplicative type, and T is diagonalizable if and only if it is split.

Let M be a group scheme of multiplicative type, which is of finite type over $\mathcal{F}$. Suppose given a morphism $f : M \to G$ over $\mathcal{F}$. Then we have:

Theorem 3.1.1. *The centralizer $H = C_G(f)$ of the image of f is a closed, smooth, reductive $\mathcal{F}$-subgroup scheme of G.*

Proof. This result is originally due to Richardson; see **[richardson82:MR651417]**. It can also be obtained as follows: **[demazure70:MR0274459]** shows that H is a closed and smooth subgroup scheme [4]. Since M is of multiplicative type, $M_{\overline{\mathcal{F}}}$ is diagonalizable, where $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$. It follows from **[jantzen03:MR2015057]** that all linear representations of $M_{\overline{\mathcal{F}}}$ are completely reducible; now **[conrad15:MR3362817]** shows that the identity component of the centralizer H of M is indeed reductive. □

It is more straightforward to see that the the centralizer of the image of f is reductive when the image of f lies in a maximal torus. Indeed, we have the following:

Proposition 3.1.2. *Suppose that f factors through a maximal torus T of G. Then:*
(a) The centralizer $H = C_G(f(M))$ of the image of f is a reductive subgroup of G.
(b) Suppose that T (and hence G) is split, write Φ for the set of roots of T in $\operatorname{Lie}(G)$, and for $\alpha \in \Phi$, write U_α for the root subgroup. Then the set of roots of H is $\Phi' = \{\alpha \in \Phi \mid f^*\alpha = 0\}$.

Proof. It suffices to prove the result after extending the ground field; thus, to prove (a) we may and will suppose that T is a split torus and that M is diagonalizable. As in the statement of (b), let $\Phi \subset X^*(T)$ be the roots of T in $\operatorname{Lie}(G)$. Write $f^* : X^*(T) \to X^*(M)$ for the mapping determined on characters by f.

Put $\Phi' = \{\alpha \in R \mid f^*\alpha = 0\}$ as in (b). Since M is diagonalizable, $\operatorname{Lie}(H) = \operatorname{Lie}(G)^M$. For $\alpha \in R$, it follows that $\operatorname{Lie}(G)_\alpha \subset \operatorname{Lie}(H)$ if and only if $\alpha \in \Phi'$.

Now, T is contained in H. Thus, in the terminology of **[demazure11:MR2867622]**, H is a subgroup of type (R). Now **[demazure11:MR2867622]** implies that H is reductive once we observe that $\Phi' = -\Phi'$. But by definition,

$$\alpha \in \Phi' \iff j^*(\alpha) = 0 \iff -\alpha \in \Phi',$$

whence the Theorem. □

3.2. **The group scheme μ_n.** Fix an integer $n \geq 1$ and let μ_n be the group scheme with coordinate ring $\mathcal{F}[\mu_n] = \mathcal{F}[T]/\langle T^n - 1\rangle$. We may view μ_n as the scheme theoretic kernel of the mapping $(t \mapsto t^n) : \mathbf{G}_m \to \mathbf{G}_m$, or alternately as the *Cartier dual* $\mu_n = (\mathbf{Z}/n\mathbf{Z})^D$ of the cyclic group $\mathbf{Z}/n\mathbf{Z}$ viewed as a "constant group scheme" over $\mathcal{F}$ – see **[knus98:MR1632779]**. It is a finite and commutative group scheme over $\mathcal{F}$, and it is smooth over $\mathcal{F}$ if and only if n is invertible in $\mathcal{F}$; see e.g. Example (21.11) and Example (21.5)(4) of **[knus98:MR1632779]**.

Let $n, m \in \mathbf{Z}_{>0}$ with $m \mid n$ and consider the homomorphisms

$$\mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \quad \text{via} \quad (a + m\mathbf{Z} \mapsto a \cdot \frac{n}{m} + n\mathbf{Z})$$

and

$$\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \quad \text{via} \quad (a + n\mathbf{Z} \mapsto a + m\mathbf{Z}).$$

[4] In fact, G need not be reductive for this claim; H is closed and smooth provided only that G is smooth

Via Cartier duality, these mappings respectively induce homomorphisms of group schemes

$$\tau_{n,m} : \mu_n \to \mu_m \quad \text{and} \quad \iota_{m,n} : \mu_m \to \mu_n.$$

It follows from the formalism of Cartier duality that $\iota_{m,n}$ is a closed embedding, and that $\ker(\tau_{n,m}) \simeq \mu_{n/m}$ is the image of $\iota_{n/m,n}$.

3.3. **The infinitesimal group scheme μ_q with q a power of p.** Suppose now that the characteristic of $\mathcal{F}$ is $p > 0$, and let $q = p^s$ be a power of p for some $s \geq 1$. In this section, we investigate homomorphisms $\phi : \mu_q \to H$ for a smooth affine $\mathcal{F}$-group scheme H.

We first recall the following:

Proposition 3.3.1. *[**demazure11:MR2867622**] Let M be a connected and reductive group over $\mathcal{F}$. Then the center $Z(M)$ of M (as a group scheme) is contained in each maximal torus of M.*

The main result we require is the following *folk result*:

Proposition 3.3.2. *If $\phi : \mu_q \to G$ is a homomorphism of group schemes, the image of ϕ is contained in a torus of G.*

We now deduce the result in two different ways; see also Remark 3.3.6 for further discussion.

First proof of Proposition 3.3.2. It is enough to argue that the image of ϕ is contained in *some* torus in G. Let $M = C^o_G(\phi)$ be the identity component of the centralizer of the image of ϕ. According to Theorem 3.1.1, one knows that M is reductive. Since μ_q is connected, evidently the image of ϕ is contained in the *center* of the connected and reductive group M. Now the result follows from Proposition 3.3.1, and the proof is complete. □

We now outline a proof that *avoids* use of the conclusion from Theorem 3.1.1 that the centralizer M is reductive. To this end, we will first establish that when $s = 1$, such a homomorphism $\phi : \mu_p \to H$ is completely determined by its tangent mapping.

Recall that a finite dimensional p-Lie algebra L over $\mathcal{F}$ has a restricted enveloping algebra $U^{[p]}(\mathrm{L})$ which is a co-commutative Hopf algebra over $\mathcal{F}$ with dimension $p^{\dim L}$ as $\mathcal{F}$-vector space.

Proposition 3.3.3. *The coordinate algebra $\mathcal{F}[\mu_p]$ is isomorphic to the* dual *Hopf algebra of $U^{[p]}(\mathrm{Lie}(\mu_p))$.*

Proof. This follows from **[demazure11:MR2867621]**.

Since the argument is straightforward, for the benefit of the reader we present the following sketch. Write $\mathfrak{m} \subset \mathcal{F}[\mu_p]$ for the *augmentation ideal* – i.e. the kernel of the augmentation mapping $\mathcal{F}[\mu_p] \to \mathcal{F}$. As explained in **[jantzen03:MR2015057]**, the algebra of distributions $\mathrm{Dist}(\mu_p)$ consists of all linear functionals $\mu \in \mathcal{F}[\mu_p]^\vee = \mathrm{Hom}_{\mathcal{F}}(\mathcal{F}[\mu_p], \mathcal{F})$ for which μ vanishes on some power of $\mathfrak{m}$; since the ideal $\mathfrak{m}$ is nilpotent, $\mathrm{Dist}(\mu_p)$ coincides with the dual Hopf algebra $\mathcal{F}[\mu_p]^\vee$. Now, the Lie algebra $\mathrm{Lie}(\mu)$ identifies with the space of linear functionals μ for which $\mu(1) = 0$ and $\mu(\mathfrak{m}^2) = 0$. The inclusion $\mathrm{Lie}(\mu_p) \to \mathrm{Dist}(\mu_p)$ induces an algebra homomorphism $U^{[p]}(\mathrm{Lie}(\mu_p)) \to \mathrm{Dist}(\mu_p)$, and it is straightforward to see that this mapping is injective. Since $\dim_{\mathcal{F}} U^{[p]}(\mathrm{Lie}(\mu_p)) = p = \dim_{\mathcal{F}} \mathcal{F}[\mu_p]$, the conclusion of the Proposition follows. □

We are now going to produce a basis vector $\delta = \delta_1$ for $\mathrm{Lie}(\mu_p)$, as follows. Note that $\mathcal{F}[\mu_p] = \mathcal{F}[T]/\langle T^p - 1\rangle = \mathcal{F}[t]$. The elements $(t-1)^i$ form a $\mathcal{F}$-basis for $\mathcal{F}[\mu_p]$ for $0 \leq i \leq p-1$. For each $0 \leq i \leq p-1$, there is an element $\delta_i \in \mathcal{F}[\mu_p]^\vee$ for which

$$\langle \delta_i, (t-1)^j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

and $\{\delta_i\}$ form a $\mathcal{F}$-basis for $\mathcal{F}[\mu_p]^\vee$. In fact, δ_0 is the mapping $f \mapsto f(1)$, and δ_1 is the "point-derivation" given by $f \mapsto \left(t\dfrac{df}{dt}\right)|_{t=1}$; the element $\delta = \delta_1$ spans the Lie algebra $\mathrm{Lie}(\mu_p)$ as a $\mathcal{F}$-vector space, and a simple calculation shows that $\delta_1^{[p]} = \delta_1$.

Proposition 3.3.4. *Let H be an affine group scheme over $\mathcal{F}$. Then the assignment $\phi \mapsto d\phi(\delta_1)$ determines a bijection*

$$\mathrm{Mor}_{\text{gp scheme}}(\mu_p, H) \to \{Y \in \mathrm{Lie}(H) \mid Y = Y^{[p]}\}$$

where $\mathrm{Mor}_{\text{gp scheme}}(\mu_p, H)$ denotes the set of homomorphisms of $\mathcal{F}$-group schemes from μ_p to H.

Proof. This follows from the result proved in **[demazure11:MR2867621]** and also in **[demazure11:MR2867621]**; see **[conrad15:MR3362817]**, as well.

However, in the relatively simple situation at hand, we can give a simple proof, as follows. First, since H and μ_p are affine, the group scheme homomorphisms $\mu_p \to H$ are in one-to-one correspondence with the Hopf algebra homomorphisms $\mathcal{F}[H] \to \mathcal{F}[\mu_p]$.

Let $\mathfrak{p} \subset \mathcal{F}[H]$ and $\mathfrak{m} \subset \mathcal{F}[\mu_p]$ be the kernels of the respective augmentation mapping. Write $I = \langle f^p \mid f \in \mathfrak{p}\rangle \subset \mathcal{F}[H]$. It follows from **[jantzen03:MR2015057]** that $\mathcal{F}[H_1] = \mathcal{F}[H]/I$ is a Hopf algebra which determines an infinitesimal subgroup scheme $H_1 \subset H$.

Since $\mathfrak{m}^p = 0$, any Hopf algebra homomorphism $\mathcal{F}[H] \to \mathcal{F}[\mu_p]$ must vanish on I and hence factors through a Hopf algebra homomorphism $\mathcal{F}[H_1] = \mathcal{F}[H]/I \to \mathcal{F}[\mu_p]$. It follows from **[jantzen03:MR2015057]** that $\mathcal{F}[H_1]$ identifies with the dual Hopf algebra to $U^{[p]}(\mathrm{Lie}(H))$.

Now, in view of Proposition 3.3.3, taking duals gives a natural bijection between Hopf algebra mappings

$$U^{[p]}(\mathrm{Lie}(\mu_p)) \to U^{[p]}(\mathrm{Lie}(H))$$

and Hopf algebra mappings

$$U^{[p]}(\mathrm{Lie}(H))^\vee = \mathcal{F}[H_1] \to U^{[p]}(\mathrm{Lie}(\mu_p))^\vee = \mathcal{F}[\mu_p].$$

Finally, an algebra homomorphism $U^{[p]}(\mathrm{Lie}(\mu_p)) \to U^{[p]}(\mathrm{Lie}(H))$ is completely determined by a homomorphism of p-Lie algebras $\mathrm{Lie}(\mu_p)) \to \mathrm{Lie}(H)$; since δ_1 is a basis vector for $\mathrm{Lie}(\mu_p)$, such homomorphisms of p-Lie algebras correspond bijectively with those elements $Y \in \mathrm{Lie}(H)$ for which $Y^{[p]} = Y$. ☐

Proposition 3.3.5. *(a) If $1 \le m < n$, the tangent mapping $d\tau_{p^n,p^m} : \mathrm{Lie}(\mu_{p^n}) \to \mathrm{Lie}(\mu_{p^m})$ is zero.*
(b) If $1 \le m \le n$, the tangent mapping $d\iota_{p^m,p^n} : \mathrm{Lie}(\mu_{p^m}) \to \mathrm{Lie}(\mu_{p^n})$ is an isomorphism.
(c) Suppose that H is a smooth and affine $\mathcal{F}$-group scheme, that $\phi : \mu_{p^m} \to H$ is a homomorphism of group schemes, and that $d\phi = 0$. Then ϕ factors through the homomorphism $\tau_{p^m,p^{m-1}} : \mu_{p^m} \to \mu_{p^{m-1}}$.

Proof. We first observe that for all $n \ge 1$, the inclusion $\mu_{p^n} \subset \mathbf{G}_m$ induces an isomorphism

$$\mathrm{Lie}(\mu_{p^n}) \to \mathrm{Lie}(\mathbf{G}_m).$$

Now, the mapping $\mu_{p^n} \to \mu_{p^m}$ in (a) is induced by the mapping $t \mapsto t^{p^{n-m}} : \mathbf{G}_m \to \mathbf{G}_m$, whose tangent mapping is indeed zero. Moreover, the mapping $\mu_{p^m} \to \mu_{p^n}$ in (b) is induced by the *identity mapping* $\mathbf{G}_m \to \mathbf{G}_m$, whose tangent mapping is indeed an isomorphism.

As to (c), recall that we may identify μ_p as the image of ι_{p,p^m} in μ_{p^m}, and as the kernel of the mapping $\tau_{p^m,p^{m-1}}$. Since $d\phi = 0$, it follows from Proposition 3.3.4 that the restriction of ϕ to μ_p is trivial, hence ϕ factors through the quotient $\mu_{p^m}/\mu_p \simeq \mu_{p^{m-1}}$, as required. ☐

We can now give the

Second proof of Proposition 3.3.2. Writing $q = p^s$, we are going to show by induction on $s \ge 1$ that the image of ϕ is contained in a maximal torus of G.

Let us first treat the case $s = 1$. Let $\phi : \mu_p \to G$ be a homomorphism of group schemes. Let $X = d\phi(\delta) \in \mathrm{Lie}(G)$. Since $X^{[p]} = X$, we see for each linear representation (ρ, V) of G that the minimal polynomial for the action of $d\rho(X)$ on V divides the separable polynomial $T^p - T$, and **[springer98:MR2458469]** shows that X is a *semisimple* element of $\mathrm{Lie}(G)$.

Moreover **[borel91:MR1102012]** show that there is an $\mathcal{F}$-torus $T \subset G$ with $X \in \mathrm{Lie}(T)$. It now follows from Proposition 3.3.4 that the image of ϕ is contained in T, as required. This completes the proof when $p = q$.

We remark that we now know $M = C^o_G(\phi)$ to be a reductive subgroup containing a maximal torus of G when $q = p$, using the more elementary result Proposition 3.1.2.

Now suppose that $q = p^s$ for $s > 1$ and that the result is known by induction for homomorphisms μ_{p^t} with $t < s$. In view of Proposition 3.3.5(c), we may suppose that $d\phi \neq 0$.

Now let $X_0 = d\phi(\delta_1)$ where $\delta_1 \in \mathrm{Lie}(\mu_p) = \mathrm{Lie}(\mu_q)$ is the basis element fixed in the remarks preceding Proposition 3.3.4. According to that proposition, X_0 determines a homomorphism $\phi_0 : \mu_p \to G$ which clearly centralizes the image of ϕ. By the induction hypothesis (or just the case $q = p$), we find a maximal torus T of G containing the image of ϕ_0. Now let $M_0 = C^o_G(\phi_0)$; we've remarked already that M_0 is a reductive subgroup of G. Since T and μ_q are connected, M_0 contains T and the image of ϕ. Moreover, the image of ϕ_0 is contained in the center Z_0 of M_0.

Since $X_0 \in \mathrm{Lie}(Z_0)$, the composite homomorphism

$$\tilde{\phi} : \mu_q \xrightarrow{\phi} M_0 \to M_0/Z_0$$

has $d\tilde{\phi} = 0$, so Proposition 3.3.5 implies that $\tilde{\phi}$ factors through the homomorphism $\tau_{p^m,p^{m-1}}$. It now follows by induction on m that $\tilde{\phi}$ takes values in a maximal torus S of M_0/Z_0. But then the pre-image in M_0 of a maximal torus of M_0/Z_0 is a maximal torus of M_0 - and hence of G - containing the image of ϕ, and the proof of the Theorem is complete. □

Remark 3.3.6. (a) After completing an initial version of this manuscript, I learned that Brian Conrad recently gave a proof of Proposition 3.3.2 – see the appendix to [**martens-thaddeus-varGrothendieck**]; the argument given in the "Second proof of Proposition 3.3.2" is similar to the one given by Conrad.

(b) In an email communication in 2007, J-P. Serre communicated to me a proof of Proposition 3.3.2 similar to the above "second proof".

(c) Still another proof of the proposition is given in the recent paper [**pepin-lehalleur15:MR3525844**].

3.4. **μ-homomorphisms with values in a reductive group.** In this section, we introduce the notion of a μ-homomorphism. We begin by extending Proposition 3.3.2 to cover all μ_n. More precisely:

Proposition 3.4.1. *Let $n \in \mathbf{Z}_{\geq 1}$. If $\phi : \mu_n \to G$ is a homomorphism of group schemes, then the image of ϕ lies in a torus of G.*

Proof. Write $n = q \cdot n_0$ where $q = p^m$ is a power of the characteristic p, and where $\gcd(p, n_0) = 1$. Write $\phi_0 = \phi_{|\mu_q}$ and $\phi_1 = \phi_{|\mu_{n_0}}$. We have seen in Proposition 3.3.2 that the image of ϕ_0 is contained in a maximal torus of G; in particular Proposition 3.1.2 shows that $M_0 = C^o_G(\phi_0)$ is reductive. Proposition 3.3.1 now shows that the image of ϕ_0 is contained in each maximal torus of M_0. Thus the theorem will follow provided we prove that the image of ϕ_1 lies in a maximal torus of M_0.

It is therefore enough to prove the Theorem in the case of a homomorphism $\phi : \mu_n \to G$ where n satisfies $\gcd(n, p) = 1$. Let $E \supset \mathcal{F}$ be a Galois extension containing a primitive n-th root of unity ζ. It follows from [**springer98:MR2458469**] that $s = \phi(\zeta)$ lies in a maximal torus of G_E. Then $C^o_{G_E}(s)$ is reductive by Proposition 3.1.2. In particular, the image of ϕ_E lies in each maximal torus of $C^o_{G_E}(s)$. Since $C^o_{G_E}(s) = (C^o_G(\phi))_E$, it follows that the image of ϕ is contained in each maximal torus of $C^o_G(\phi)$, and this completes the proof. □

Continue to suppose that G is a reductive group over $\mathcal{F}$. Given homomorphisms $\phi : \mu_n \to G$ and $\psi : \mu_m \to G$, we regard ϕ and ψ as *equivalent* provided that for some $d \in \mathbf{Z}$ with $m \mid d$ and $n \mid d$, the mappings $\phi \circ \tau_{d,n}$ and $\psi \circ \tau_{d,m}$ coincide.

Definition 3.4.2. By a μ-homomorphism with values in a reductive group G, we mean an equivalence class of a homomorphism $\phi_0 : \mu_m \to G$ of group schemes. We will denote such an equivalence class symbolically by $\phi : \mu \to G$.

The following is an immediate consequence of Proposition 3.4.1:

Corollary 3.4.3. *If $\phi : \mu \to G$ is a μ-homomorphism, there is a maximal torus T of G such that any homomorphism $\mu_n \to G$ representing ϕ has image in T.*

For each integer N, we identify $\mathbf{Z}/N\mathbf{Z}$ with the subgroup $\dfrac{1}{N}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$ by the assignment $a + N\mathbf{Z} \mapsto \dfrac{a}{N} + \mathbf{Z}$. If T is a split torus, we write $\mathrm{Hom}(\mu, T)$ for the group of all μ-homomorphisms with values in T. The next result describes this group.

Proposition 3.4.4. *Let T be a split torus. The assignment*

$$\mathrm{Hom}(\mu, T) \to \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Q}/\mathbf{Z}) \simeq Y \otimes \mathbf{Q}/\mathbf{Z}$$

given by $\phi \mapsto \phi_0^*$ is bijective, where $\phi_0 : \mu_n \to T$ represents ϕ, where $\phi_0^* : X \to \mathbf{Z}/n\mathbf{Z} = \dfrac{1}{n}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$ is the map on character groups that ϕ_0 determines via Cartier duality, where $X = X^*(T)$ is the character group of T, and where $Y = X_*(T) \simeq X^\vee$ is the cocharacter group of T.

Proof. Let $\phi, \psi : \mu \to T$ be μ-homomorphisms, represented respectively by the homomorphism $\phi_0 : \mu_n \to T$ and $\psi_0 : \mu_m \to T$. Recall that $\tau^*_{an,n} : \mathbf{Z}/n\mathbf{Z} = \frac{1}{n}\mathbf{Z}/\mathbf{Z} \to \mathbf{Z}/an\mathbf{Z} = \frac{1}{an}\mathbf{Z}/\mathbf{Z}$ is given by $\dfrac{t}{n} + \mathbf{Z} \mapsto \dfrac{at}{an} + \mathbf{Z}$. It is thus easy to see that ϕ_0 and ψ_0 are equivalent if and only ϕ_0^* and ϕ_1^* coincide as homomorphisms $X \to \mathbf{Q}/\mathbf{Z}$; this shows that the indicated mapping is well-defined and injective.

If now $f : X \to \mathbf{Q}/\mathbf{Z}$ is any group homomorphism, the image of f is finitely generated hence contained in the subgroup $\dfrac{1}{N}\mathbf{Z}/\mathbf{Z} = \mathbf{Z}/N\mathbf{Z}$ for some N. Then f determines a homomorphism $\phi_f : \mu_N \to T$ via Cartier duality, and it is clear that the above mapping assigns f to the μ-homomorphism represented by ϕ_f; thus the indicated mapping is surjective. □

Remark 3.4.5. Let $\mathcal{A}$ denote an integral domain with field of fractions K, and let T be a split torus over K. Write $\mathscr{T}$ for the *canonical* $\mathcal{A}$-group scheme associated to T as in [**bruhat84:MR756316**]. Thus $\mathscr{T}$ is a split torus over $\mathcal{A}$ and $X^*(\mathscr{T}) = X^*(T)$. Using Proposition 3.4.4, the discussion in [**bruhat84:MR756316**] shows that the assignment $\phi \mapsto \phi_0^*$ determines a bijection between the collection $\mathrm{Hom}_{\mathcal{A}}(\mu, \mathscr{T})$of μ-homomorphisms $\phi : \mu_{\mathcal{A}} \to \mathscr{T}$ over $\mathcal{A}$ and the group $\mathrm{Hom}_{\mathbf{Z}}(X^*(T), \mathbf{QZ})$

If ϕ is a μ-homomorphism with values in the split torus T, we write $\phi^* = \phi_0^*$ for the corresponding element of $Y \otimes \mathbf{Q}/\mathbf{Z}$, with notation as in Proposition 3.4.4. Conversely, if $x \in V = Y \otimes \mathbf{Q}$ and $\overline{x} \in Y \otimes \mathbf{Q}/\mathbf{Z}$ the image of x, we write $\phi_x = \phi_{\overline{x}}$ for the corresponding homomorphism $\mu \to T$.

We now give a description of the centralizer of a μ-homomorphism with values in a maximal split torus of G, and thus describes the subgroups M of G of type $C(\mu)$ as discussed in the introduction to this paper.

Theorem 3.4.6. *Assume that G is split reductive over $\mathcal{F}$, and write T for a maximal split $\mathcal{F}$-torus. Suppose that ψ be a μ-homomorphism with values in T, and write $M = C_G^o(\psi)$.*

(a) M is a reductive subgroup of G containing the maximal torus T.

(b) Let $\Phi \subset X^*(T)$ be the set of roots of G with respect to T, and let $Y = X_*(T)$ be the cocharacter group of T. For $\alpha \in \Phi$, write U_α for the corresponding root subgroup of G. Choose $y \in Y \otimes \mathbf{Q}$ representing the element $\psi^* \in Y \otimes \mathbf{Q}/\mathbf{Z}$, and let Φ_y denote the corresponding closed and symmetric system of roots

$$\Phi_y = \{\alpha \in \Phi \mid \langle \alpha, y \rangle \in \mathbf{Z}\}$$

as in section 2.5. Then

$$M = \langle T, U_\alpha \mid \alpha \in \Phi_y \rangle.$$

In particular, Φ_y is the root system of M.

Proof. (a) just restates proposition 3.4.1(b).

Now for $\alpha \in \Phi$, it is easy to see that μ acts trivially on U_α if and only if $\langle \alpha, x \rangle \in \mathbf{Z}$. Thus the characterization of M in (b) follows immediately from Proposition 3.1.2. □

Remark 3.4.7. (a) The Theorem now shows that section 2.5 describes the root systems Φ_x for the connected centralizer M of the image of a μ-homomorphism. In particular, Theorem 2.5.3 describes a simple system of roots for Φ_x, and thus Remark 2.5.4 describes the Dynkin diagram of M.

(b) The root system Φ_y is of course independent of the choice of representative $y \in Y \otimes \mathbf{Q}$ representing $\psi^* \in Y \otimes \mathbf{Q}/\mathbf{Z}$. Moreover, the above description shows that subgroup M only depends on the (W_{aff}-orbit of the) *facet* of V containing y.

(c) Following **[Oberwolfach-Serre]**, one can describe μ-homomorphisms using *Kac coordinates*, as follows. Suppose that G is simple and that $\{\alpha_1, \dots, \alpha_r\}$ is a system of simple roots. After replacing $y \in Y \otimes \mathbf{Q}$ by an W_{aff}-conjugate point and thus we may suppose - in the terminology and notation of section 2.5 - that y is contained in the $\mathbf{Q}$-simplex with vertices $0 = \varpi_0, \varpi_1/n_1, \dots, \varpi_r/n_r$. In particular, $y = \sum_{i=1}^r a_i \varpi_i / n_i$ for certain $a_i \in \mathbf{Q}$ with $0 \le a_i \le 1$ for each i, and $0 \le \sum a_i \le 1$. The rational numbers $(a_1, \dots, a_r)$ are known as the Kac coordinates of the μ-homomorphism.

If $\mathcal{F}$ is algebraically closed, the Kac coordinates determine the μ-homomorphism up to conjugation in $G(\mathcal{F})$; see **[pepin-lehalleur15:MR3525844]**.

3.5. **Some automorphisms of subgroups of type** $C(\mu)$**.** Keep the notations of section 3.4; thus G is a split reductive group over $\mathcal{F}$, T is a maximal split torus of G, $X = X^*(T)$, and $Y = X_*(T)$.

Fix $y \in Y \otimes \mathbf{Q}$, and let $\phi = \phi_y : \mu \to T$ be the μ-homomorphism determined by $\overline{y} \in Y \otimes (\mathbf{Q}/\mathbf{Z})$. As before, consider the identity component $M = M_y$ of the centralizer in G of the image of ϕ_y.

Let Z denote the center of M, and write $M_{\text{ad}} = M/Z$ for the *adjoint quotient* of M. Then the image $T_1 = T/Z$ of T in M_{ad} is a maximal torus of M_{ad}.

Proposition 3.5.1. *(a) The character group of T_1 is given by $X^*(T_1) = \mathbf{Z}\Phi_y$.*
(b) y determines a cocharacter $\psi_y \in X_*(T_1)$ with

$$\langle \gamma, \psi_y \rangle = \langle \gamma, y \rangle \quad \text{for } \gamma \in \Phi_y.$$

(c) For any $t \in \mathcal{F}^\times$, $t_y = \operatorname{Int}(\psi_y(t))$ determines a $\mathcal{F}$-automorphism of M. Moreover, if $\alpha \in \Phi_y$ and $x_\alpha : \mathbf{G}_a \to U_\alpha$ is any isomorphism such that $\operatorname{Ad}(s)x_\alpha(u) = x_\alpha(\alpha(s)u)$ for $s \in T(\mathcal{F})$ and $u \in \mathcal{F}$, then $t_y x_\alpha(u) = x_\alpha(t^{\langle \alpha, y \rangle} u)$ for $u \in \mathcal{F}$

Proof. Indeed, (a) follows since M/Z is an adjoint semisimple group with root system Φ_y. Since $X_*(T_1)$ is dual to $X^*(T_1) = \mathbf{Z}\Phi_y$, the existence of the cocharacter ψ_y in (b) follows from (a). Now (c) follows from definitions. ☐

3.6. **Examples of subgroups of type** $C(\mu)$**.** In this section, we present some examples to illustrate some features of subgroups of type $C(\mu)$. We begin by pointing out that there are subgroups of type $C(\mu)$ which are not the centralizer of any semisimple element of G. Note the following:

Proposition 3.6.1. *Assume that G is $\mathcal{F}$-split, absolutely simple of adjoint type with split maximal torus T, let $\alpha_1, \dots, \alpha_\ell \in \Phi \subset X^*(T)$ be a system of simple roots, let $\varpi_i^\vee \in X_*(T)$ be the fundamental dominant co-roots, and let $\widetilde{\alpha}$ be the highest root. Write*

$$\widetilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i \quad \text{for } n_i \in \mathbf{Z}_{>0}.$$

For $1 \le i \le \ell$, $\varpi_i/n_i \in Y \otimes \mathbf{Q}$ determines a μ-homomorphism, and we write M_i for its connected centralizer. If $n_i = p$, the center of M_i is an infinitesimal group scheme, so that M_i is not the connected centralizer of any semisimple element of G.

Proof. For each i, the root system of M_i with respect to T has $\Delta_i = \{\alpha_0 = -\widetilde{\alpha}\} \cup \{\alpha_j \mid j \ne i\}$ as a system of simple roots. It follows from **[springer70:MR0268192]** that if n_i is prime, $\mathbf{Z}\Delta/\mathbf{Z}\Delta_i \simeq \mathbf{Z}/n_i\mathbf{Z}$. Now the Proposition follows from the fact that the center of M_i is the $\mathcal{F}$-group scheme which is the *Cartier dual* – see Section 3.2 – of $\mathbf{Z}\Delta/\mathbf{Z}\Delta_i$. ☐

Remark 3.6.2. Keep the notations of Proposition 3.6.1.

(a) When $\Phi = G_2$, then $\widetilde{\alpha} = 3\alpha_1 + 2\alpha_2$. And indeed, the group M_1 with root system A_2, has infinitesimal center in characteristic 3, and the group M_2 with root system $A_1 \times A_1$ has infinitesimal center in characteristic 2.
(b) When $\Phi = E_8$, the coefficient of α_5 in $\widetilde{\alpha}$ is 5. The subgroup M_5 with root system $A_4 \times A_4$ indeed has infinitesimal center in characteristic 5.

We now demonstrate that a reductive subgroup H of G may have the property that H_E is of type $C(\mu)$ for some finite separable field extension, but that H is not of type $C(\mu)$.

Proposition 3.6.3. *Let $G = \mathrm{Sp}_4(\mathcal{F})$ be the split symplectic group of rank 2. There is a reductive $\mathcal{F}$-subgroup $H \subset G$ containing a maximal torus of G with the following properties:*
(a) H is an $\mathcal{F}$-form of $\mathrm{SL}_2 \times \mathrm{SL}_2$.
(b) There is a separable quadratic field extension E such that H_E is of type $C(\mu)$.
(c) H is not the centralizer of the image of any homomorphism $\mu \to G$ defined over $\mathcal{F}$.

Sketch. Let $\mathcal{F} \subset E$ be a separable quadratic field extension. and write $x \mapsto \overline{x}$ for the action of the non-trivial element of $\mathrm{Gal}(E/\mathcal{F})$. Consider the alternating form β on the $\mathcal{F}$-vector space $V = E \oplus E$ given by the formula $\beta(v, w) = \mathrm{tr}_{E/\mathcal{F}}(v_1\overline{w}_2 - w_1\overline{v}_2)$ where $\mathrm{tr}_{E/\mathcal{F}}$ is the trace mapping. Since E is a separable extension of $\mathcal{F}$, the alternating pairing β is non-degenerate.

Thus we may identify G with $\mathrm{Sp}(V, \beta)$. If we write $A = \mathrm{End}_{\mathcal{F}}(V) \simeq \mathrm{Mat}_4(\mathcal{F})$, there is a symplectic involution σ on A determined by the property $\beta(Xv, w) = \beta(v, \sigma(X)w)$ for $v, w \in V$; then $G = \mathrm{Iso}(A, \sigma)$ and we may describe G "functorially" by the rule

$$G(\Lambda) = \{X \in A \otimes_{\mathcal{F}} \Lambda \mid X \cdot \sigma(X) = 1\}$$

for a commutative $\mathcal{F}$-algebra Λ.

Viewing V as an E-vector space, we find the $\mathcal{F}$-subalgebra $B = \mathrm{End}_E(V) \subset A = \mathrm{End}_{\mathcal{F}}(V)$. Evidently $B \simeq \mathrm{Mat}_2(E)$ and one readily checks that B is σ-invariant; in fact, $\sigma_{|B}$ is the "standard" involution of the quaternion E-algebra B. The algebra B determines a subgroup H of G defined functorially by the rule

$$H(\Lambda) = \{b \in B \otimes_{\mathcal{F}} \Lambda \mid b \cdot \sigma(b) = 1\} \subset G(\Lambda);$$

thus $H = \mathrm{Iso}(B, \sigma_{|B})$ is a semisimple subgroup of G; moreover, $H \simeq R_{E/\mathcal{F}}\mathrm{SL}_2$ is a $\mathcal{F}$-form of $\mathrm{SL}_2 \times \mathrm{SL}_2$. This proves (a)

Now, the center of H_E is $\mu_2 \times \mu_2$. Since $V_E = V \otimes_{\mathcal{F}} E$ is the direct sum of two copies of the "natural" four dimensional representation of $\mathrm{SL}_2 \times \mathrm{SL}_2$ and is a faithful representation of G_E, it follows that H_E is the centralizer of any homomorphism $\mu \to Z(H_E) = \mu_2 \times \mu_2$ whose image is not central in G. There are precisely two such μ-homomorphisms, and they are interchanged by the action of the non-trivial element of the Galois group $\Gamma = \mathrm{Gal}(E/\mathcal{F})$. □

Remark 3.6.4. With $H \subset G$ as in the proof of Proposition 3.6.3, the given arguments show also the following:

- If the characteristic of $\mathcal{F}$ is not 2, then H is the centralizer of a semisimple element in $G(E)$.
- If the characteristic of $\mathcal{F}$ is 2, the H is the centralizer of a semisimple element

$$X \in \mathrm{Lie}(G)(E) = \mathrm{Lie}(G_E) = \mathrm{Lie}(G) \otimes_{\mathcal{F}} E,$$

  and if $Z = Z(H)$ denotes the center of H, then the group of points $Z(\mathcal{F}_{\mathrm{alg}})$ is trivial where $\mathcal{F}_{\mathrm{alg}}$ is an algebraic closure of $\mathcal{F}$.

We finally demonstrate that the property that a reductive subgroup $H \subset G$ is of type $C(\mu)$ is in general affected by isogeny.

Proposition 3.6.5. *Keep the notations of Proposition 3.6.3, let $G_1 = \mathrm{PSp}_{4,\mathcal{F}}$, and write*

$$\pi : G = \mathrm{Sp}_4 \to G_1 = \mathrm{PSp}_4$$

for the isogeny. There is a semisimple subgroup $H \subset G$ such that H is not of type $C(\mu)$ while $H_1 = \pi(H) \subset G_1$ is of type $C(\mu)$.

Sketch. Let H be as in the proof of Proposition 3.6.3. The center of $H = R_{E/\mathcal{F}}\,\mathrm{SL}_2$ is $R_{E/\mathcal{F}}\mu_2$, which is an $\mathcal{F}$-form of $\mu_2 \times \mu_2$. In fact, $\mathcal{F}$-homomorphisms $\mu_2 \to R_{E/\mathcal{F}}\mu_2$ correspond bijectively to $\mathrm{Gal}(E/\mathcal{F})$-equivariant maps $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ where $\mathrm{Gal}(E/\mathcal{F})$ acts trivially on the target, and the non-trivial element of $\mathrm{Gal}(E/\mathcal{F})$ acts on the domain group by switching the factors.

Thus there are precisely two $\mathcal{F}$-homomorphisms $\mu_2 \to R_{E/\mathcal{F}}\mu_2$, and the non-trivial such homomorphism corresponds to the inclusion of the center of $G = \mathrm{Sp}_4$ in the center of $H = R_{E/\mathcal{F}}\,\mathrm{SL}_2$. It follows that the center of H_1 is $\mathcal{F}$-isomorphic to $(R_{E/\mathcal{F}}\mu_2/\mu_2) \simeq \mu_2$. Since H – and hence also H_1 – is the connected centralizer of its center, the result follows. □

4. Reductive groups over a local field

Throughout this section, $\mathcal{A}$ will denote a complete discrete valuation ring with field of fractions K and residue field k. Consdier a connected and reductive algebraic group G over K. We are going to consider the parahoric group schemes $\mathcal{P}$ attached to G. In particular, $\mathcal{P}$ is a smooth, affine $\mathcal{A}$-group scheme, and its generic fiber $\mathcal{P}_{\mathrm{K}}$ coincides with the given group G.

In [**bruhat84:MR756316**], the authors describe the parahoric group schemes for quasisplit G using *schematic root data*; we begin by recalling this notion in Section 4.1 and the associated construction in Theorem 4.3.1. In case G is split over K with maximal split torus T, those parahoric group schemes $\mathcal{P} = \mathcal{P}_x$ which "contain T" (in a suitable sense) arise from points x in $V = X_*(T) \otimes \mathbf{Q}$; we recall this description in Section 4.3.

Suppose that G is split with root system $\Phi \subset X^*(T)$, and consider the parahoric group scheme $\mathcal{P}_x$ corresponding to $x \in V$. Then x determines a subsystem Φ_x as in Section 2.5; in Theorem 4.4.2 we show that Φ_x determines a "Chevalley schematic root datum" for the subgroup $M = C_G^o(\phi_{\overline{x}})$ of type $C(\mu)$ where $\phi_{\overline{x}}$ is the μ-homomorphism determined by the image $\overline{x}$ of x in $X_*(T) \otimes \mathbf{Q}/\mathbf{Z}$; see Theorem 3.4.6. In particular, we find a Chevalley group scheme $\mathscr{M}$ with generic fiber M, and we see $\mathscr{M}$ as a closed subscheme of $\mathcal{P}_x$.

Existence of this subgroup scheme $\mathscr{M}$ settles the proof of our main result – Theorem 1 – in case G is split. Finally, when G splits over an unramified extension, the result is obtained by descent in Section 4.5.

4.1. **A Chevalley system for a split reductive group.** Suppose that the group G is split over K. Fix a maximal split torus T of G, and let $(X, Y, \Phi, \Phi^\vee)$ denote the root datum of G with respect to T.

Thus $\Phi \subset X = X^*(T)$ is the set of roots of T in $\mathrm{Lie}(G)$. For each $\alpha \in \Phi$, there is a corresponding 1-dimensional K-subgroup U_α - the *root subgroup* - normalized by T. Write $G_\alpha = \langle U_{\pm\alpha} \rangle$.

Let us fix an isomorphism between the diagonal maximal torus S of SL_2 and the 1-dimensional split torus $\mathbf{G}_m$, and write $U^+ \subset \mathrm{SL}_2$ for the upper triangular unipotent subgroup. For a root α, a *pinning* of α will mean a central K-isogeny $\psi_\alpha : \mathrm{SL}_2 \to G_\alpha$ such that:

(P1) ψ_α maps S to T and the restriction of ψ_α to S identifies with the co-root $\alpha^\vee \in X_*(T)$ (for the fixed identification of S and $\mathbf{G}_m$), and
(P2) ψ_α maps U^+ isomorphically to U_α.

We fix an identification $U^+ \simeq \mathbf{G}_a$. Then upon restriction to the root subgroups of SL_2, ψ_α determines K-isomorphisms $\psi_{\alpha,\pm} : \mathbf{G}_a \to U_{\pm\alpha}$.

For a fixed system of simple roots $\Delta \subset \Phi$, a pinning of G relative to T is a collection of pinnings $(\phi_\alpha)_{\alpha\in\Delta}$.

Proposition 4.1.1. *[***demazure11:MR2867622***] A pinning $(\psi_\alpha)_{\alpha\in\Delta}$ of G relative to T determines a* Chevalley system $(\psi_\alpha)_{\alpha\in\Phi^+}$ *for G prolonging it.*

Remark 4.1.2. (a) Recall [**bruhat84:MR756316**] that a Chevalley system is a collection of pinnings (ψ_α) for each root $\alpha \in \Phi^+$ satisfying an additional "compatibility" property spelled out e.g. in [**bruhat84:MR756316**]].
(b) See the discussion following [**conrad15:MR3362817**] to see that the definitions we have given of "pinning" and "Chevalley system" encode the same information as the definitions given in [**demazure11:MR2867622**].

(c) If $(\psi_\alpha)_{\alpha\in\Phi^+}$ is a Chevalley system, then $(X_\alpha = d\psi_\alpha(1))_\alpha$ is a *Chevalley basis* for $\mathrm{Lie}(G)$; see e.g. **[humphreys78:MR499562]** for the definition.

4.2. **Schematic root data for a split reductive group** G**.** Following **[bruhat84:MR756316]**, we consider the notion of a *schematic root datum*

$$\mathscr{D} = (\mathscr{T}, (\mathscr{U}_a)_{a\in\Phi})$$

for G. In *loc. cit.* the case of quasi-split G is considered; we have simplified the definition somewhat since we only consider this notion when G is split over K.

According to **[bruhat84:MR756316]**, there is an essentially unique $\mathcal{A}$-split torus $\mathscr{T}$ with generic fiber $T = \mathscr{T}_{\mathrm{K}}$. By a *schematic root datum*, we mean a collection $(\mathscr{U}_a)_{a\in\Phi}$ where $\mathscr{U}_a$ is a $\mathcal{A}$-group scheme for each $a \in \Phi$, such that

($\diamondsuit$1) $\mathscr{U}_a$ is affine, flat and of finite type over $\mathcal{A}$ for each a,
($\diamondsuit$2) $U_a = \mathscr{U}_{a,\mathrm{K}}$ for each a,
($\diamondsuit$3) for $a, b \in \Phi$ with $a \neq -b$, for a suitable ordering on $\Phi \cap (a, b)$ the commutator mapping

$$\gamma_{a,b} : U_a \times U_b \to \prod_{c\in\Phi\cap(a,b)} U_c$$

arises by base change from an $\mathcal{A}$-morphism $\mathscr{U}_a \times \mathscr{U}_b \to \prod_{c\in\Phi\cap(a,b)} \mathscr{U}_c$.
($\diamondsuit$4) for $a \in \Phi$, the mapping $((z, u) \to zuz^{-1}) : T \times U_a \to U_a$ arises by base change from an $\mathcal{A}$-morphism $\mathscr{T} \times \mathscr{U}_a \to \mathscr{U}_a$.

Remark 4.2.1. In condition ($\diamondsuit$3) we have been vague about the notion "suitable ordering"; see **[bruhat84:MR756316]** for details. In this paper, we will not have occasion to verify the conditions ($\diamondsuit$1)–($\diamondsuit$4) directly.

A schematic root data $\mathscr{D}$ determines a group scheme with generic fiber G, according to the following important result:

Theorem 4.2.2. *[***bruhat84:MR756316***] Let $\mathscr{D} = (\mathscr{U}_a)_{a\in\Phi}$ be a schematic root datum for G, and suppose that all $\mathscr{U}_a$ are smooth over $\mathcal{A}$. Then there is an group scheme $\mathcal{P} = \mathcal{P}_{\mathscr{D}}$ which is affine, smooth and of finite type over $\mathcal{A}$ with connected fibers such that the following hold:*

(S1) the inclusion $T \to G$ (resp. $U_a \to G$ for $a \in \Phi$) prolongs to a isomorphism from $\mathscr{T}$ (resp. $\mathscr{U}_a$) to a closed $\mathcal{A}$-subgroup scheme of $\mathcal{P}$;
(S2) for each system of positive roots Φ^+ of Φ and for each order of Φ^+ (resp. of $\Phi^- = -\Phi^+$), the product mapping is an isomorphism of schemes from $\prod_{a\in\Phi^+} \mathscr{U}_a$ to a closed subgroup scheme $\mathscr{U}^+$ (resp. $\mathscr{U}^-$) of $\mathcal{P}$;
(S3) The product mapping determines an isomorphism of schemes from $\mathscr{U}^- \times \mathscr{T} \times \mathscr{U}^+$ to an open sub-scheme of $\mathcal{P}$.

Moreover, $\mathcal{P}$ is up to isomorphism the unique $\mathcal{A}$-group scheme with generic fiber $G = \mathcal{P}_{\mathrm{K}}$ having the properties (S1), (S2) and (S3).

Let us fix a root $\alpha \in \Phi$, and consider the rank 1 split reductive K-subgroup $G_\alpha = \langle T, U_{\pm\alpha}\rangle$. We observe:

Proposition 4.2.3. *If $\mathscr{D}$ is a schematic root datum for G as above, then $\mathscr{D}_\alpha = (\mathscr{U}_\alpha, \mathscr{U}_{-\alpha})$ is a schematic root datum for G_α. If $\mathcal{P}_\alpha$ is the $\mathcal{A}$-group scheme determined by this schematic root datum, then $\mathcal{P}_\alpha$ is a closed subgroup scheme of $\mathcal{P}$.*

Proof. It is immediate from definitions that $\mathscr{D}_\alpha$ is a schematic root datum. It only remains to argue that $\mathcal{P}_\alpha$ is a closed subgroup scheme of $\mathcal{P}$. For this, we choose a faithful linear representation $\mathscr{L}$ of $\mathcal{P}$, where $\mathscr{L}$ is a free $\mathcal{A}$-module of finite rank; see **[bruhat84:MR756316]**.

Since $\mathscr{L}$ is a faithful $\mathcal{P}$-module, it is clear that $\mathscr{L}$ is a $\mathscr{D}$-module in the sense of **[bruhat84:MR756316]** and hence also $\mathscr{L}$ is a $\mathscr{D}_\alpha$-module. We apply **[bruhat84:MR756316]** to find a closed embedding $\mathcal{P}_\alpha$ as a subgroup scheme of $\mathrm{GL}(\mathscr{L})$. Now our assertion follows since it is immediate that this embedding factors through the embedding $\mathcal{P} \subset \mathrm{GL}(\mathscr{L})$. $\square$

4.3. **Schematic root data and parahoric group schemes.** In this section, we first describe how one obtains split reductive groups over $\mathcal{A}$ using Chevalley schematic root data, essentially constructed from a Chevalley system for G as in Proposition 4.1.1 and Remark 4.1.2.

The parahoric group schemes are paramterized by points in the affine building associated with G. Since the affine building is the union of apartments each determined by a maximal split torus of G and since each maximal split torus is conjugate by an element of $G(\mathrm{K})$, it suffices to describe the parahoric subgroups associated with such an apartment. In the remainder of this section, we describe the groups schemes.

Recall that a finitely generated free $\mathcal{A}$-module M determines an $\mathcal{A}$-group scheme M_{add} whose generic fiber $M_{\mathrm{add},\mathrm{K}}$ is the vector group determined by the K-vector space $M \otimes_{\mathcal{A}} \mathrm{K}$.

Let U be a 1 dimensional commutative unipotent group scheme over K, and let $\mathbf{G}_a \xrightarrow{\psi} U$ be a fixed isomorphism. For $n \in \mathbf{Z}$, consider the fractional ideal $I = \pi^n \mathcal{A} \subset \mathrm{K}$. Then I_{add} is a smooth $\mathcal{A}$-group scheme with generic fiber $\mathbf{G}_{a\mathrm{K}}$, and by transport of structure along ψ, one finds a smooth $\mathcal{A}$-group scheme $\mathscr{U}_n$ with an identification $\mathscr{U}_{n,\mathrm{K}} = U$. Moreover, the isomorphism ψ determines an isomorphism $I = \pi^n \mathcal{A} \xrightarrow{\psi} \mathscr{U}_n(\mathcal{A})$. Compare **[bruhat84:MR756316]**

More generally, for $r \in \mathbf{Q}$, write $\lceil r \rceil = \min\{n \in \mathbf{Z} \mid n \geq r\}$ for the "ceiling function", and write $\mathscr{U}_r = \mathscr{U}_{\lceil r \rceil}$. Thus $\mathscr{U}_r(\mathcal{A}) = \{\psi(a) \mid \nu(a) \geq r\}$ where $\nu : \mathrm{K}^\times \to \mathbf{Z}$ is the (normalized) valuation of K, and $\mathscr{U}_r = (\pi^{\lceil r \rceil} \mathcal{A})_{\mathrm{add}}$. [5]

Consider now a Chevalley system (ψ_α) for the group G and the split torus T, as in Proposition proposition 4.1.1. For each $\alpha \in \Phi$, recall that ψ_α determines isomorphisms $\psi_{\alpha,\pm} : \mathbf{G}_a \to U_{\pm\alpha}$. We write $\mathscr{U}_{\pm\alpha} = \mathscr{U}_{\pm\alpha,0}$ for the $\mathcal{A}$-group schemes obtained as above by transport of structure along $\psi_{\alpha,\pm}$ from the unit ideal $\mathcal{A}$.

Theorem 4.3.1. *(a) The collection $(\mathscr{U}_\alpha)_{\alpha\in\Phi}$ determines a schematic root datum $\mathscr{D}_{\mathrm{Chev}(\Phi)}$, called a **Chevalley schematic root datum**.*
(b) The group scheme $\mathcal{G} = \mathcal{P}_{\mathscr{D}}$ determined by the Chevalley schematic root datum $\mathscr{D} = \mathscr{D}_{\mathrm{Chev}(\Phi)}$ as in Theorem 4.2.2 is a split reductive group scheme over $\mathcal{A}$.

Proof. The result amounts to the description of a split reductive group scheme over $\mathcal{A}$, and thus essentially follows from [**demazure11:MR2867622**]. Using the language of schematic root data, (a) follows from [**bruhat84:MR756316**], and (b) follows from [**bruhat84:MR756316**]. □

We immediately obtain:

Proposition 4.3.2. *Let $\alpha \in \Phi$ be a root, and let $\mathcal{G}_\alpha$ be the group scheme determined by the schematic root datum $(\mathscr{U}_\alpha, \mathscr{U}_{-\alpha})$ as in Proposition 4.2.3. Then $\mathcal{G}_\alpha \subset \mathcal{G}$ is a split reductive group with $\mathcal{G}_{\alpha,\mathrm{K}} = G_\alpha$.*

Now recall that we write $V = X_*(T) \otimes \mathbf{Q}$. We point out that $V \otimes_{\mathbf{Q}} \mathbf{R}$ is the apartment of the affine building of G determined by T. We are going to describe the parahoric group schemes determined by points in V.

Fix $x \in V$. For $\alpha \in \Phi$, consider the group scheme $\mathscr{U}_{\alpha,x} = (\mathscr{U}_\alpha)_{\langle\alpha,x\rangle}$. Thus $\mathscr{U}_{\alpha,x}$ is obtained via $\psi_{\alpha,+}$ from the $\mathcal{A}$-group scheme I_{add} where I denotes the ideal $\pi^{\lceil\langle\alpha,x\rangle\rceil}\mathcal{A}$.

Theorem 4.3.3. *The collection $\mathscr{D}_x = (\mathscr{U}_{\alpha,x})_{\alpha\in\Phi}$ is a schematic root datum.*

Proof. Use [**bruhat84:MR756316**] to see that the function $\Phi \to \mathbf{R}$ given by $\alpha \mapsto \langle\alpha, x\rangle$ is *concave*. Now the assertion follows from [**bruhat84:MR756316**]. □

We write $\mathcal{P} = \mathcal{P}_x$ for the group scheme obtained via Theorem 4.2.2 from the schematic root datum $\mathscr{D}_x$ for G. For a facet F in V, write $\mathcal{P}_F = \mathcal{P}_x$ for some (any) point $x \in F$.

Definition 4.3.4. When G is K-split, the *parahoric group schemes* attached to G are the $G(\mathrm{K})$-conjugates of the group schemes $\mathcal{P}_F$ of Theorem 4.3.3 for F a facet in V.

[5]Of course, the notation $\mathscr{U}_r$ can be made meaningful for $r \in \mathbf{R}$, but we only require this notion for rational r.

Remark 4.3.5. We have assumed $G = \mathcal{P}_{\mathrm{K}}$ to be connected. Since G is split, the special fiber $\mathcal{P}_{\mathrm{k}}$ is always connected; see Theorem 4.2.2.

In [**bruhat84:MR756316**], the *parahoric subgroups* of $G(\mathrm{K})$ are the "connected stabilizers" in $G(\mathrm{K})$ of facets of the *building* of G. It follows from [**bruhat84:MR756316**], the parahoric subgroups are precisely the $G(\mathrm{K})$-conjugates of the subgroups $\mathcal{P}_x(\mathcal{A})$ for some x as above. See also [**bruhat84:MR756316**]. Note that from the point of view of the action of $G(\mathrm{K})$ on its affine building, when G is simply connected, the subgroup $\mathcal{P}_x(\mathcal{A})$ is precisely the stabilizer of the point x, but not in general – see e.g. the example $G = \mathrm{PGL}_n$ in [**tits79:MR546588**].

The main results of this paper apply only to reductive groups G which split over an unramified extension of K. As we note below in Proposition 4.5.2, the parahoric group schemes in this setting – see Definition 4.5.3 – are obtained by descent from the split case. On the other hand, for a quasisplit group G which splits only over a ramified extension of K, in general a group scheme arising from the appropriate notion of schematic root datum for G need not have connected fibers; we don't consider these group schemes in this manuscript.

In Section 4.4, we are going to obtain a Levi decomposition of the special fiber of $\mathcal{P}_x$. In order to do so, we require results about the reductive quotient of this special fiber. Recall that we introduced in section 2.5 the subsystem Φ_x. Here is an alternative description which will be useful below.

Lemma 4.3.6. $\Phi_x = \{\alpha \in \Phi \mid \lceil\langle\alpha, x\rangle\rceil + \lceil\langle-\alpha, x\rangle\rceil = 0\}$.

Proof. Indeed, for any rational (or real) number r, $\lceil r\rceil + \lceil -r\rceil = 0$ if and only if $r \in \mathbf{Z}$. $\square$

Proposition 4.3.7. *(a) The unipotent radical of the special fiber $\mathcal{P}_{x,\mathrm{k}}$ – a linear algebraic* k*-group – is defined and split over* k*.*

(b) The reductive quotient of the special fiber $\mathcal{P}_{x,\mathrm{k}}$ of $\mathcal{P}$ is a split reductive group over k *with root system Φ_x.*

Proof. For (a), note first that if k is *perfect*, the unipotent radical of *any* linear algebraic k-group – and in particular, that of $\mathcal{P}_{\mathrm{k}}$ – is defined and split over k; see e.g. [**springer98:MR2458469**]. Thus for the proof of (a) we may and will suppose that the characteristic of k is $p > 0$.

To proceed, we first suppose that the characteristic of K is also $p > 0$. In particular, there is an inclusion $\mathbf{F}_p \subset \mathcal{A}$ where $\mathbf{F}_p$ denotes the field with p elements. Let $\varpi \in \mathcal{A}$ be a uniformizer; in particular, ϖ is transcendental over $\mathbf{F}_p$. Since $\mathcal{A}$ is complete, the inclusion $\mathbf{F}_p[\varpi]_{\langle\varpi\rangle} \subset \mathcal{A}$ prolongs to an inclusion of the completion $\mathcal{B} = \mathbf{F}_p[\![\varpi]\!]$ in $\mathcal{A}$. Writing $L = \mathbf{F}_p(\!(\varpi)\!)$ for the field of fractions of $\mathcal{B}$, we find an embedding $L \subset \mathrm{K}$.

Now consider the split reductive group H over L with the same root datum as G; cf. e.g. the *Existence Theorem* found in [**conrad15:MR3362817**]. We may and will identify the split reductive K-groups G and H_{K}.

With the identification of the preceding paragraph, the torus T is - up to $G(\mathrm{K})$-conjugacy - obtained by base change from a split maximal L-torus T' of H. Identifying $V = X_*(T)$ with $V' = X_*(T')$, we denote by $\mathcal{Q}$ the parahoric $\mathcal{B}$-group scheme with $\mathcal{Q}_L = H$ determined by (the point in V' corresponding to) x. It is then clear that $\mathcal{P} = \mathcal{Q}_{\mathcal{A}}$ - i.e. that $\mathcal{P}$ arises by base change from $\mathcal{Q}$. Since $\mathbf{F}_p$ is the residue field of $\mathcal{B}$, it follows that $\mathcal{P}_{\mathrm{k}}$ arises by extension of scalars from $\mathcal{Q}_{\mathbf{F}_p}$. Since $\mathbf{F}_p$ is *perfect*, the unipotent radical of $\mathcal{Q}_{\mathbf{F}_p}$ is defined and split over $\mathbf{F}_p$ and the result now follows.

If instead K has characteristic zero, one must make a different choice of $\mathcal{B}$. In this case, it follows from [**serre79:MR554237**] that there is an injective homomorphism $W(\mathrm{k}) \to \mathcal{A}$ where $W(\mathrm{k})$ denotes the ring of *Witt vectors* having residue field k. Setting $\mathcal{B} = W(\mathbf{F}_p) = \mathbf{Z}_p$, functoriality of the construction of Witt vectors yields a canonical mapping $\mathcal{B} \to W(\mathrm{k})$. Now one argues as before to see that $\mathcal{P}$ arises by base change from a smooth group scheme $\mathcal{Q}$ over $\mathcal{B}$; since the residue field of $\mathcal{B}$ is perfect, this completes the proof of (a).

For (b), combine Lemma 4.3.6 with [**bruhat84:MR756316**]. $\square$

Remark 4.3.8. In fact, the previous result remains valid when $\mathcal{A}$ is only Henselian rather than complete. The proofs of (b) and (c) require no change; in the proof of (a), one instead takes for $\mathcal{B}$ either the

Henselization of $\mathbf{F}_p[\varpi]_{\langle\varpi\rangle}$ in the equal characteristic case, or the Henselization of $\mathbf{Z}_{\langle p\rangle}$ in the mixed-characteristic case.

Proposition 4.3.9. *If $x, x' \in V$ and $x - x' \in X_*(T) \subset V$, then $\mathscr{D}_{x'}$ is obtained from $\mathscr{D}_x$ via the inner automorphism $\mathrm{Int}(h)$ for some $h \in G(\mathrm{K})$. In particular, h determines an isomorphism of $\mathcal{A}$-group schemes $\mathcal{P}_{x'} \simeq \mathcal{P}_x$*

Proof. $\phi = x - x' \in X_*(T) \subset V$. Then $\phi : \mathbf{G}_m \to T$ is a cocharacter. For each $\alpha \in \Phi$, the element $h = \phi(\pi) \in T(\mathrm{K})$ satisfies

$$\mathrm{Int}(h)\mathscr{U}_{\alpha,x'} = \mathrm{Int}(\phi(\pi))\mathscr{U}_{\alpha,x'} = \mathscr{U}_{\alpha,x'+\phi} = \mathscr{U}_{\alpha,x}.$$

Thus indeed $\mathscr{D}_{x'} = \mathrm{Int}(h)\mathscr{D}_x$. Now the isomorphism $\mathcal{P}_{x'} \simeq \mathcal{P}_x$ follows from the uniqueness in Theorem 4.2.2. ☐

Proposition 4.3.10. *Suppose that the split reductive G has semisimple rank 1, let T be a split maximal torus of G, and write $\alpha, -\alpha$ for the roots. Let $x \in X_*(T) \otimes \mathbf{Q}$. After choosing a Chevalley system, x determines a schematic root datum $(\mathscr{U}_{\alpha,x}, \mathscr{U}_{-\alpha,x})$ as in Theorem 4.3.3 and thus an $\mathcal{A}$-group scheme $\mathcal{P} = \mathcal{P}_x$ as in Theorem 4.2.2. Suppose that*

$$\lceil\langle\alpha, x\rangle\rceil + \lceil\langle-\alpha, x\rangle\rceil = 0$$

Then $\mathcal{P} = \mathcal{P}_x$ is (split) reductive over $\mathcal{A}$.

Proof. Consider the reductive K-group $H = G/Z$ where Z is the center of G. Then H is a rank 1 adjoint group. Consider the 1-dimensional split maximal torus $S = T/Z \subset H$. Of course, the roots $\pm\alpha \in X^*(T)$ are in the image of the natural mapping $X^*(S) \to X^*(T)$. There is a cocharacter $\phi \in X_*(S)$ for which $\langle\alpha, \phi\rangle = 1$; in fact $X_*(S) = \mathbf{Z}\phi$.

Now, the action of G on itself by inner automorphisms determines an action of H on G. Write $r = \lceil\langle\alpha, x\rangle\rceil$. The automorphism of G determined by the element $\phi(\pi^r) \in S(\mathrm{K}) \subset H(\mathrm{K})$ yields $\mathcal{A}$-isomorphisms $\mathscr{U}_{\pm\alpha} \xrightarrow{\sim} \mathscr{U}_{\pm\alpha,x}$, hence this automorphism of G determines an isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{P}$ of $\mathcal{A}$-group schemes. ☐

4.4. **Reductive subgroup schemes of a parahoric group scheme for split G.** We keep the assumptions and notations of the Section 4.3; in particular, G is a split reductive group over K with split maximal torus T, and $x \in V = X_*(T) \otimes \mathbf{Q}$.

Our goal in this section is to prove the validity of the conclusion of Theorem 1 for the split group G. Thus, we must exhibit a suitable reductive subgroup scheme of the parahoric group scheme $\mathcal{P}_x$ which was described in Section 4.3.

We are going to require the following result which provides a condition for a linear algebraic group to be reductive. To state the result, consider any field $\mathcal{F}$ and let H be a linear algebraic group over $\mathcal{F}$. Let $T \subset H$ be a non-trivial $\mathbf{F}$-split torus, let $\Delta \subset X^*(T)$ be a linearly independent subset, and for each $\alpha \in \Delta$ suppose that there is a reductive $\mathcal{F}$-subgroup $H_\alpha \subset H$ containing T as a maximal torus and having roots $\alpha, -\alpha$. Write $V_{\pm\alpha}$ for the root subgroups of H_α.

Proposition 4.4.1. *Suppose for each pair $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ that $V_{-\alpha}$ and V_β commute, and that $H = \langle H_\alpha \mid \alpha \in \Delta\rangle$. Then H is reductive.*

Proof. Our formulation is essentially that found in [**steinberg99:MR1694546**], though Steinberg works there over an algebraically closed base field. The result in the required generality is a consequence of [**conrad15:MR3362817**]. ☐

We now return to the study of the parahoric group scheme $\mathcal{P} = \mathcal{P}_x$. Recall that the image of x in $X_*(T) \otimes \mathbf{Q}/\mathbf{Z} = Y \otimes \mathbf{Q}/\mathbf{Z}$ determines a μ-homomorphism with values in T as in section 3.4. In fact, as in Remark 3.4.5, the class of x determines a μ-homomorphism $\phi_x : \mu_{\mathcal{A}} \to \mathscr{T}$ which is defined over $\mathcal{A}$, where $\mathscr{T}$ is the split $\mathcal{A}$-torus with generic fiber T used in the construction of $\mathcal{P}_x$.

Let $M = C^0_G(\phi_{x,\mathrm{K}})$ denote the identity component of the centralizer in G of the image of the μ-homomorphism $\phi_{x,\mathrm{K}}$; thus M is a subgroup of G of type $C(\mu)$ described in Theorem 3.4.6. We now consider the scheme-theoretic centralizer $C_{\mathcal{P}_x}(\phi_x)$ of the image of ϕ_x, and - as in [**bruhat84:MR756316**]

- we consider the identity component $\mathscr{M} = C^0_{\mathcal{P}_x}(\phi_x)$ of this centralizer. Thus the fibers $\mathscr{M}_{\mathrm{K}}$ and $\mathscr{M}_{\mathrm{k}}$ are connected linear algebraic groups over K resp. k.

We are going to prove:

Theorem 4.4.2. *Let $\mathcal{P} = \mathcal{P}_x$.*

(a) $\mathscr{M}$ is a locally closed subgroup scheme of $\mathcal{P}$ which is smooth over $\mathcal{A}$ and has generic fiber $\mathscr{M}_{\mathrm{K}} = M$.
(b) $\mathscr{M}$ is a split reductive $\mathcal{A}$-group scheme.
(c) The special fiber $\mathscr{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$.

Proof. In (a), the assertion that $\mathscr{M}_{\mathrm{K}} = M$ is immediate from definitions. Since $\mu_{n,\mathcal{A}}$ is a diagonalizable group scheme, [**demazure11:MR2867621**] shows the centralizer $C_{\mathcal{G}}(\phi_x)$ to be a closed subgroup scheme of $\mathcal{P}_x$ which is smooth over $\mathcal{A}$. Now, according to [**bruhat84:MR756316**], the identity component $\mathscr{M}$ is an open subgroup scheme of this centralizer, so indeed $\mathscr{M}$ is smooth over $\mathcal{A}$ and locally closed in $\mathcal{P}_x$; (a) is now proved.

Since $\mathscr{M}$ is smooth over $\mathcal{A}$ with reductive generic fiber $\mathscr{M}_{\mathrm{K}}$, (b) will follow if we argue that the special fiber $\mathscr{M}_{\mathrm{k}}$ is reductive.

Fix a pinning $(\psi_\alpha)_{\alpha\in\Delta}$ for G and hence a Chevalley schematic root datum $\mathscr{D}_{\mathrm{Chev}(\Phi)} = (\mathscr{U}_\alpha)_{\alpha\in\Phi}$ as in Theorem 4.3.1. Now let $\mathscr{D}_x = (\mathscr{U}_{\alpha,x})_{\alpha\in\Phi}$ be the schematic root datum obtained from $\mathscr{D}_{\mathrm{Chev}(\Phi)}$ using x, as in Theorem 4.3.3, so that $\mathcal{P}$ is the parahoric group scheme determined by $\mathscr{D}_x$, as in section 4.3.

Recall that according to Proposition 4.2.3, the schematic root datum $\mathscr{U}_{\alpha,x}, \mathscr{U}_{-\alpha,x})$ determines a closed $\mathcal{A}$-subgroup scheme $\mathcal{P}_\alpha$ of $\mathcal{P}$. When $\alpha \in \Phi_x$ it follows from Lemma 4.3.6 together with Proposition 4.3.10 that the subgroup scheme $\mathcal{P}_\alpha$ is $\mathcal{A}$-reductive. Moreover, it clear from the construction that $\mathcal{P}_\alpha$ is centralized by ϕ_x, so that $\mathcal{P}_\alpha$ is contained in $\mathscr{M}$.

Fix a basis Δ for the root system Φ_x – see e.g. Theorem 2.5.3. Now let $H \subset \mathscr{M}_{\mathrm{k}}$ be the k-subgroup generated by the subgroups $\mathcal{P}_{\alpha,\mathrm{k}}$ for $\alpha \in \Delta$. Writing $\mathscr{U}_{\pm\alpha} \subset \mathcal{P}_\alpha$ for the subgroup schemes determined by our schematic root data, one knows that $\mathscr{U}_\alpha$ commutes with $\mathscr{U}_{-\beta}$ for $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$; indeed, it suffices to observe that these subgroups commute on the generic fiber.

Now Proposition 4.4.1 implies that the subgroup $H \subset \mathscr{M}_{\mathrm{k}}$ is reductive. Now write π for the quotient mapping from the k-group $\mathcal{P}_{\mathrm{k}}$ to its reductive quotient $\mathcal{P}_{\mathrm{k,red}}$. It is clear for $\beta \in \Phi_x$ that π maps $\mathscr{U}_{\beta,\mathrm{k}}$ onto the corresponding root subgroup of $\mathcal{P}_{\mathrm{k,red}}$. According to Proposition 4.3.7, $\mathcal{P}_{\mathrm{k,red}}$ has root system Φ_x. It follows that π maps H *onto* $\mathcal{P}_{\mathrm{k,red}}$. Since $\ker\pi$ is unipotent, it follows that H is a Levi factor of $\mathcal{P}_{\mathrm{k}}$ and in particular H is *isomorphic* to $\mathcal{P}_{\mathrm{k,red}}$. In particular, the dimension of H coincides with the dimension of the generic fiber $\mathscr{M}_{\mathrm{K}}$. Since $\mathscr{M}$ is smooth with connected fibers, and since $H \subset \mathscr{M}_{\mathrm{k}}$, it follows that $H = \mathscr{M}_{\mathrm{k}}$; (b) and (c) are now proved. ☐

Remark 4.4.3. (a) In the proof of Theorem 4.4.2, one can actually avoid the use of Proposition 4.4.1 as follows.

With notation as in the Theorem, let $M = C_G(\phi_{x,\mathrm{K}})$. One first argues that $\mathscr{D}_{x,M} = (\mathscr{U}_{\alpha,x})_{\alpha\in\Phi_x}$ is a schematic root datum for M. By an argument like that in Proposition 4.3.10, one now argues that $\mathscr{D}_{x,M}$ is conjugate by an element of $T(\mathrm{K})$ to a Chevalley schematic root datum for M. Thus $\mathscr{D}_{x,M}$ determines a reductive subgroup scheme $\mathcal{Q}$ of $\mathcal{P} = \mathcal{P}_x$. Evidently $\mathcal{Q}$ is centralized by ϕ_x. It remains to argue that $\mathcal{Q}$ coincides with the centralizer $\mathscr{M}$ subgroup scheme; this holds essentially for dimension reasons.

(b) I thank Gopal Prasad for communicating to me the suggestion to use the result Proposition 4.4.1 to prove that the centralizer $\mathscr{M}$ in Theorem 4.4.2 is reductive.

(c) As a referee pointed out to me, the preceding Theorem can be nicely stated using the *affine building* of G. Namely, the parahoric group scheme $\mathcal{P}$ is determined by a point x in the building. Now, each *apartment* D of the building which contains x determines a split $\mathcal{A}$-torus $\mathscr{T}_D$ in $\mathcal{P}$ – see [**bruhat84:MR756316**]. The point x determines a μ-homomorphism $\phi_x : \mu \to \mathscr{T}_D$ and the connected centralizer of the image of ϕ_x yields a reductive subgroup scheme $\mathscr{M}_D \subset \mathcal{P}$ determined by D.

4.5. **Reductive subgroup schemes of a parahoric group scheme.** Now suppose that G splits over an unramified, separable field extension of K. The following result provides more precise information:

Proposition 4.5.1. *[**bruhat84:MR756316**] There is a finite, unramified, Galois extension* L *of* K*, a maximal* K*-split torus S of G and a maximal* K*-torus T of G containing S for which T_{L} is* L*-split.*

Fix an extension L as in the preceding proposition, and write $\mathcal{B}$ for the integral closure of $\mathcal{A}$ in L. Let us fix a maximal K-split torus S. Also write $\Gamma = \mathrm{Gal}(\mathrm{L}/\mathrm{K})$. For any maximal K-torus T, note that Γ acts on the cocharacter $X_*(T_{\mathrm{L}})$ and hence on $V_{\mathrm{L}} = X_*(T_{\mathrm{L}}) \otimes \mathbf{Q}$.

Certain parahoric group schemes for G_{L} *descend* to $\mathcal{A}$-group schemes, as follows:

Proposition 4.5.2 (Étale descent)**.** *Let T be a maximal* K*-torus of G which contains S and splits over* L*, and let $V_{\mathrm{L}} = X_*(T_{\mathrm{L}}) \otimes \mathbf{Q}$. Let F be a Γ-invariant facet in V_{L}, and let $\mathcal{Q} = \mathcal{Q}_F = \mathcal{Q}_x$ be the corresponding parahoric group scheme for G_{L} determined by any point $x \in F$. There is a smooth, affine $\mathcal{A}$-group scheme $\mathcal{P}$ with connected fibers such that*

(i) generic fiber $\mathcal{P}_{\mathrm{K}}$ may be identified with G, and
(ii) $\mathcal{Q} \simeq \mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}$.

Proof. Since $\mathrm{K} \subset L$ is unramified, this follows from [**bruhat84:MR756316**]. □

Definition 4.5.3. Assume that G splits over an unramified extension L of K. The *parahoric group schemes* attached to G are the $G(\mathrm{K})$-conjugates of group schemes $\mathcal{P}$ obtained by étale descent – see Proposition 4.5.2 – from parahoric group schemes $\mathcal{Q} = \mathcal{Q}_F$ determined by a Γ-stable facet $F \subset X_*(T_{\mathrm{L}}) \otimes \mathbf{Q}$ for some maximal K-torus T of G containing S which splits over L.

Remark 4.5.4. Let $\mathcal{P}$ be a parahoric group scheme for G which arise by étale descent as in Proposition 4.5.2. Thus $\mathcal{P}_{\mathcal{B}} = \mathcal{Q}_F$ for some Γ-stable facet F in $V_{\mathrm{L}} = X_*(T) \otimes \mathbf{Q}$. It follows from [**bruhat84:MR756316**] that there is a point $x \in F \cap X_*(S) \otimes \mathbf{Q}$ – i.e. a point in F fixed by the action of Γ. Thus $\mathcal{Q}_F = \mathcal{Q}_x$.

The analogue of Proposition 4.3.7(a) remains valid for parahoric group schemes in this more general setting, as follows:

Proposition 4.5.5. *Let $\mathcal{P} = \mathcal{P}_x$ be a parahoric group scheme with generic fiber $G = \mathcal{P}_{\mathrm{K}}$, and suppose that G splits over an unramified extension of* K*. Then the unipotent radical of the linear algebraic* k*-group $\mathcal{P}_{\mathrm{k}}$ is defined and split over* k*.*

Proof. Indeed, in view of Proposition 4.3.7(a), the Proposition follows from the following more general statement, which may be deduced from [**springer98:MR2458469**]: Suppose that $\mathrm{k} \subset \ell$ is a separable field extension and that H is a linear algebraic k-group, and assume moreover that the unipotent radical of H is defined and split over ℓ. Then the unipotent radical of H is defined and split over k. □

We now prove Theorem 1 from the introduction. Recall the statement:

Theorem 4.5.6. *Assume that G splits over an unramified extension of* K*. There is a reductive $\mathcal{A}$-subgroup scheme $\mathscr{M} \subset \mathcal{P}$ containing $\mathscr{T}$ such that*

(a) the special fiber $\mathcal{P}_{\mathrm{k}}$ has a Levi decomposition with Levi factor $\mathscr{M}_{\mathrm{k}}$, and
(b) the generic fiber $\mathscr{M}_{\mathrm{K}}$ is a connected reductive subgroup of G containing T, and $\mathscr{M}_{\mathrm{L}}$ is a subgroup of G_{L} of type $C(\mu)$.

Proof. In case G is already split over K, the Theorem is an immediate consequence of Theorem 4.4.2. Otherwise, using Remark 4.5.4 the parahoric group scheme $\mathcal{P}$ arises via étale descent from the $\mathcal{B}$-group scheme $\mathcal{Q}_x$ for some point $x \in X_*(S) \otimes \mathbf{Q} \subset X_*(T) \otimes \mathbf{Q}$; thus x is Γ-stable.

Via étale descent, one finds $\mathcal{A}$-group schemes $\mathscr{S} \subset \mathscr{T}$ for which $\mathscr{S}_{\mathrm{K}} = S$ and $\mathscr{T}_{\mathrm{K}} = T$; thus $\mathscr{S}$ is a split $\mathcal{A}$-torus, and $\mathscr{T}_{\mathcal{B}}$ is a split $\mathcal{B}$-torus. Using Remark 3.4.5, we see that the image of x in $x \in X_*(\mathscr{S}) \otimes \mathbf{Q}/\mathbf{Z}$ determines a μ-homomorphism $\phi_x : \mu \to \mathscr{S}$ over $\mathcal{A}$.

Now let $\mathscr{M}_1$ denote the $\mathcal{B}$-subgroup scheme of $\mathcal{Q}_x$ which is the centralizer of the image of the μ-homomorphism $\phi_{x,\mathcal{B}} : \mu_{\mathcal{B}} \to \mathscr{S}_{\mathcal{B}} \subset \mathscr{T}_{\mathcal{B}}$. Thus $\mathscr{M}_1$ is the subgroup scheme of $\mathcal{Q}_x$ described by Theorem 4.4.2.

Since $\mathscr{T}$ and x remain invariant under the action of the Galois group Γ, and since $\mathscr{M}_1$ is uniquely determined by $\mathscr{T}$ and x, it is clear that $\mathscr{M}_1$ remains invariant under this action; more precisely, the algebra $\mathcal{B}[\mathscr{M}_1] \subset \mathrm{L}[\mathscr{M}_{1,\mathrm{L}}]$ is invariant under the semilinear Γ-action on $\mathrm{L}[\mathscr{M}_{1,\mathrm{L}}]$.

It now folows by étale descent – see **[bruhat84:MR756316]** – that there is a smooth group scheme $\mathscr{M}$ over $\mathcal{A}$ with $\mathscr{M}_{\mathcal{B}} = \mathscr{M}_1$. The group scheme $\mathscr{M}$ has the required properties □

Remark 4.5.7. Assume that the residue field k is *perfect*. If G is an "inner form" (by which I mean: an inner form of a split reductive group K-group), then G splits over an unramified extension of K.

Indeed, suppose that G is an inner form of a split group, and write L for the maximal unramified extension contained in a some separable closure of K. Since k is perfect, the residue field of L is an algebraic closure of k. Thus, a theorem of Lang implies that L has cohomological dimension ≤ 1; see [**serre94:MR1324577**]. Now a theorem of Steinberg implies that $H^1(\mathrm{L}, H)$ is trivial for every connected linear algebra group over L; see [**serre94:MR1324577**]. In particular, it follows that every reductive L-group is *quasisplit* – i.e. contains a Borel subgroup defined over L. Since by assumption the reductive group G_{L} is an inner form of a split group, and on the other hand, a reductive group is an inner form of a unique quasisplit group – see e.g. **[knus98:MR1632779]** –, it follows that G splits over L – i.e. G splits over an unramified extension.

5. Examples

In this section, we give an example illustrating Theorem 4.5.6 in case G is not split but has an unramified splitting field; see section 5.1. And we show by example that in general the conclusion of Theorem 4.5.6 does not hold when G fails to split over an unramified extension; see section 5.3 and section 5.4.

5.1. **An inner form of** $\mathrm{SL}(V)$**.** Let $d \ge 1$, and let E be a central K-division algebra with $\dim_{\mathrm{K}} E = d^2$. Suppose that the *ramification index* e of E is given by $e = d$; this is e.g. immediate if k is *finite* – see [**serre79:MR554237**]. Write $\mathcal{E}$ for the integral closure of $\mathcal{A}$ in E; according to [**reiner03:MR1972204**], $\mathcal{E}$ is the unique maximal $\mathcal{A}$-order in E.

Suppose that there is a maximal subfield L of E which is an unramified extension of K (necessarily of degree d); if k is perfect, this assumption follows from [**serre79:MR554237**].

Now, if π_E denotes a prime element of E, then $\ell = \mathcal{E}/\pi_E\mathcal{E}$ is a commutative field and is an extension of k of degree d. Let $\mathcal{B}$ be the integral closure of $\mathcal{A}$ in L. Since $\mathrm{L} \supset \mathrm{K}$ is unramified, $\pi_E\mathcal{E} \cap \mathcal{B} = \pi\mathcal{B}$ where π is a uniformizer of $\mathcal{A}$ and hence also of $\mathcal{B}$. Now deduce that $\mathcal{B}/\pi\mathcal{B}$ embeds in $\mathcal{E}/\pi_E\mathcal{E} = \ell$, hence $\mathcal{B}/\pi\mathcal{B} \simeq \ell$ since $[\mathcal{B}/\pi\mathcal{B} : \mathrm{k}] = [\ell : \mathrm{k}]$.

Let G be the "unit group scheme" $E^\times$; thus G is a reductive group over K which is an inner form of the group GL_d; we have $G_{\mathrm{L}} \simeq \mathrm{GL}_{d,\mathrm{L}}$. Let $T \simeq R_{\mathrm{L}/\mathrm{K}}\mathbf{G}_m$ denote the maximal K-torus of G determined by the maximal subfield $\mathrm{L} \subset E$.

Since G is anisotropic modulo its center, there is a unique parahoric group scheme over $\mathcal{A}$ associated to G; it is the unit group scheme $\mathcal{P} = \mathcal{E}^\times$.

Since $\mathcal{B}$ is integral over $\mathcal{A}$, we have $\mathcal{B} \subset \mathcal{E}$. Thus $\mathscr{T} = R_{\mathcal{B}/\mathcal{A}}\mathbf{G}_m$ is a smooth $\mathcal{A}$-subgroup scheme of $\mathcal{P} = \mathcal{E}^\times$ with special fiber $\mathscr{T}_{\mathrm{k}} \simeq R_{\ell/\mathrm{k}}\mathbf{G}_m$. Since $\mathcal{E}/\pi\mathcal{E} \simeq \ell$, we conclude that the reductive quotient of the special fiber $\mathcal{P}_{\mathrm{k}}$ is isomorphic to the k-torus $\mathscr{T}_{\mathrm{k}}$.

In fact, the parahoric group scheme $\mathcal{P}$ arises by étale descent from an Iwahori group scheme $\mathcal{Q}$ over $\mathcal{B}$ with generic fiber $\mathrm{GL}_{d,\mathrm{L}}$. The reductive quotient of the special fiber $\mathcal{Q}_\ell$ is a split torus of dimension d. Evidently, the conclusion of Theorem 4.5.6 holds by taking the reductive subgroup scheme $\mathscr{M} = \mathscr{T}$, which arises by étale descent from a maximal split $\mathcal{B}$-torus of $\mathcal{Q}$.

5.2. **A Lemma.** Suppose that $\mathcal{P}$ is a smooth group scheme over $\mathcal{A}$ whose generic fiber $G = \mathcal{P}_{\mathrm{K}}$ is reductive, and suppose that $\mathscr{M}$ is a reductive subgroup scheme of $\mathcal{P}$ whose generic fiber $\mathscr{M}_{\mathrm{K}}$ is a subgroup of $\mathcal{P}_{\mathrm{K}}$ which is geometrically of type $C(\mu)$. We write $\overline{\mathrm{k}}$ and $\overline{\mathrm{K}}$ for algebraic closures of k and K.

Lemma 5.2.1. *(a) The root system of $\mathscr{M}_{\overline{\mathrm{k}}}$ identifies with the root system of $\mathscr{M}_{\overline{\mathrm{K}}}$.*

(b) *Let Φ denote the root system of $G_{\overline{\mathrm{K}}}$. Then the root system of $\mathscr{M}_{\overline{\mathrm{k}}}$ is a subsystem of Φ of the form Φ_x as in Theorem 2.5.3.*

Proof. (a) is proved in [**demazure11:MR2867622**]. Since by assumption $\mathscr{M}_{\mathrm{K}}$ is geometrically of type $C(\mu)$, the assertion of (b) follows from (a) together with Theorem 3.4.6. □

5.3. **A unitary group.** Let $G = \mathrm{SU}(V, h)$ be a quasisplit unitary group which splits over the separable quadratic extension $\mathrm{K} \subset L$ with $\dim_{\mathrm{L}} V = n$, so that the reductive K-group G is a form of SL_{2n}; in fact, $G_{\mathrm{L}} \simeq \mathrm{SL}_{2n,\mathrm{L}}$.

Invoking some observations in our earlier manuscript [**mcninch10:MR2753264**], we note the following:

Proposition 5.3.1. *Assume that $\mathrm{K} \subset L$ is a ramified extension.*

(a) *There is a parahoric group scheme $\mathcal{P}$ attached to G for which the reductive quotient of $\mathcal{P}_{\mathrm{k}}$ is isomorphic to $\mathrm{Sp}_{2n,\mathrm{k}}$, the split symplectic group of rank n over k.*
(b) *The conclusion of Theorem 1 is invalid for the parahoric group scheme $\mathcal{P}$.*

Proof. (a) follows from the description in [**mcninch10:MR2753264**].

For (b), suppose by way of contradiction that $\mathcal{H} \subset \mathcal{P}$ is a reductive subgroup scheme satisfying the conditions of Theorem 1. Then Lemma 5.2.1(a) shows that the root system Ψ of $\mathcal{H}_{\overline{\mathrm{k}}}$ identifies with that of $\mathcal{H}_{\overline{\mathrm{K}}}$.

According to the preceding Proposition, the special fiber $\mathcal{P}_{\mathrm{k}}$ is a simple k-group whose root system is of type C_n. Since $\mathcal{P}_{\mathrm{k}} \simeq \mathcal{H}_{\mathrm{k}}$, it follows that the root system Ψ is of type C_n.

By assumption $\mathcal{H}_{\mathrm{K}}$ is a reductive subgroup of G containing a maximal torus, and in particular $\mathcal{H}_{\mathrm{L}}$ is a reductive subgroup of $G_{\mathrm{L}} = \mathrm{SL}_{2n,\mathrm{L}}$ containing a maximal torus. Lemma 5.2.1(b) now shows that Ψ has the form Φ_x where Φ is the root system of G_{L} – i.e. Φ is of type A_{2n-1}.

It is clear that any subsystem Φ_x of Φ is simply laced. Since Ψ is not simply laced, we have arrived at a contradiction; this completes the proof of (b). □

5.4. **Triality** D_4**.** Let G be a simply connected, quasisplit group of type ${}^3D_4$ with splitting field L, a cubic Galois extension of K – see e.g. [**springer98:MR2458469**]. Let us suppose that L is a *ramified* (and hence *totally ramified*) extension of K; write $\mathcal{B}$ for the integral closure of $\mathcal{A}$ in L.

According to [**bruhat84:MR756316**], the choice of a Chevalley system for G_{L} determines a "Chevalley-Steinberg valuation" for G and in particular – see [**bruhat84:MR756316**] – this choice determines group schemes $\mathscr{U}_\alpha$ for each α in the set Φ of K-roots of G.

Since G is quasisplit and simply connected, the torus T is "induced"; in fact, $T \simeq R_{\mathrm{L}/\mathrm{K}}(\mathbf{G}_m) \times \mathbf{G}_m$. Thus we may take $\mathscr{T} = R_{\mathcal{B}/\mathcal{A}}(\mathbf{G}_m) \times \mathbf{G}_{m/\mathcal{A}}$ and then $\mathscr{D} = (\mathscr{T}, \mathscr{U}_\alpha)_{\alpha \in \Phi}$ is a schematic root datum; see [**bruhat84:MR756316**]. Let $\mathcal{P}$ be the parahoric group scheme determined by the schematic root datum $\mathscr{D}$ as in Theorem 4.2.2. We first observe that since $\mathrm{L} \supset \mathrm{K}$ is totally ramified, we have

$$\mathcal{B} \otimes_{\mathcal{A}} \mathrm{k} \simeq \mathrm{k}[\tau]/\langle \tau^3 \rangle;$$

it follows that a maximal torus of the special fiber of the $\mathcal{A}$-group scheme $R_{\mathcal{B}/\mathcal{A}}(\mathbf{G}_m)$ is k-split and 1 dimensional. In particular, the special fiber of the group scheme $\mathscr{T}$ is split and of dimension 2.

Proposition 5.4.1. (a) *The reductive quotient of the special fiber $\mathcal{P}_{\mathrm{k}}$ is a split simple* k*-group of type G_2.*
(b) *The conclusion of Theorem 1 is invalid for the parahoric group scheme $\mathcal{P}$.*

Proof. We've remarked already that $\mathcal{P}_{\mathrm{k}}$ has a torus which is split and of dimension 2. Since the relative root system of G is of type G_2, [**bruhat84:MR756316**] shows that the reductive quotient of $\mathcal{P}_{\mathrm{k}}$ is a split simple k-group of type G_2.

As to (b), first note that the the absolute root system Φ of G is of type D_4. Suppose there is a subgroup scheme $\mathscr{M}$ of the parahoric satisfying the conclusion of Theorem 1. According to Lemma 5.2.1, the root system Ψ has the form Φ_x where Φ is the root system of G. Since Φ is a root system of type D_4, any root system of the form Φ_x is simply laced.

But the root system Ψ of $\mathscr{M}_{\mathrm{k}}$ identifies with that of the reductive quotient of $\mathcal{P}_{\mathrm{k}}$; since a root system of type G_2 isn't simply laced, we have arrived at a contradiction. Assertion (b) now follows. □

Remark 5.4.2. Suppose $p > 2$. Let U denote the unipotent radical of $\mathcal{P}_{\mathrm{k}}$. One can argue that as a module for $\mathcal{P}_{\mathrm{k}}$, the Lie algebra $\mathrm{Lie}(U)$ has as composition factors two copies of the 7 dimensional irreducible representation V_7 for the reductive quotient group of type G_2. Since the representation V_7 is a *standard highest weight module* for G_2 – i.e. in the notation of **[jantzen03:MR2015057]**, V_7 is isomorphic to the highest weight module $H^0(\lambda)$ for some dominant weight λ – it follows from **[jantzen03:MR2015057]** that $H^i(G_2, V_7) = 0$ for $i \geq 0$. It now follows from **[mcninch10:MR2753264]** that $\mathcal{P}_{\mathrm{k}}$ indeed has a Levi decomposition that is uniquely determined up to $U(\mathrm{k})$-conjugation. The conclusion of Proposition 5.4.1 simply means that this Levi factor can't arise as the special fiber of a reductive subgroup scheme satisfying the stipulations found in Theorem 1.

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