# NILPOTENT ELEMENTS AND REDUCTIVE SUBGROUPS OVER A LOCAL FIELD

GEORGE J. MCNINCH

Abstract. Let $\mathcal{K}$ be a *local field* – i.e. the field of fractions of a complete DVR $\mathcal{A}$ whose residue field $k$ has characteristic $p > 0$ – and let $G$ be a connected, absolutely simple algebraic $\mathcal{K}$-group $G$ which splits over an unramified extension of $\mathcal{K}$. We study the rational nilpotent orbits of $G$– i.e. the orbits of $G(\mathcal{K})$ in the nilpotent elements of $\mathrm{Lie}(G)(\mathcal{K})$ – under the assumption $p > 2h-2$ where $h$ is the Coxeter number of $G$.

A reductive group $M$ over $\mathcal{K}$ is *unramified* if there is a reductive model $\mathcal{M}$ over $\mathcal{A}$ for which $M = \mathcal{M}_{\mathcal{K}}$. Our main result shows for any nilpotent element $X_1 \in \mathrm{Lie}(G)$ that there is an unramified, reductive $\mathcal{K}$-subgroup $M$ which contains a maximal torus of $G$ and for which $X_1 \in \mathrm{Lie}(M)$ is *geometrically distinguished*.

The proof uses a variation on a result of DeBacker relating the nilpotent orbits of $G$ with the nilpotent orbits of the reductive quotient of the special fiber for the various parahoric group schemes associated with $G$.

## Contents



## 1. Introduction

Let $G$ be a connected and reductive algebraic group $G$ over a field $\mathcal{F}$. If $\mathcal{F}_{alg}$ is an algebraic (or even just separable) closure, there is a central isogeny $S \times G_1 \times \cdots \times G_d \to G_{\mathcal{F}_{alg}}$ where $S$ is a torus and where each $G_i$ is quasisimple over $\mathcal{F}_{alg}$. If $R_i$ denotes the (irreducible) root system associated with $G_i$, then up to re-ordering, the list of root systems $R_1, \dots, R_d$ is independent of any choices made. We write $h$ for the supremum of the Coxeter number of the $R_i$, and throughout this paper, we refer to $h$ as "the Coxeter number of $G$".

1.1. **Nilpotent elements and their orbits.** In this paper, we consider the rational orbits of $G$ in its adjoint action on $\mathrm{Lie}(G)$ – i.e. the orbits of the group of $\mathcal{F}$-points $G(\mathcal{F})$ on the elements of $\mathrm{Lie}(G) = \mathrm{Lie}(G)(\mathcal{F})$ [1]. More precisely, we study the rational orbits of *nilpotent* elements. An element $X \in \mathrm{Lie}(G)$ is nilpotent if $d\rho(X)$ is a nilpotent endomorphism of $V$ for every algebraic representation $(\rho, V)$ of $G$. From another perspective, $X$ is nilpotent provided that the derivation of the coordinate algebra $\mathcal{F}[G]$ determined by $X$ act *locally nilpotently* on $\mathcal{F}[G]$ – see [Spr98, §2.4 and §4.4]; since $\mathscr{E}[G] = \mathcal{F}[G] \otimes_{\mathcal{F}} \mathscr{E}$, this makes clear that $X \in \mathrm{Lie}(G)$ is nilpotent if and only if $X$ is nilpotent in $\mathrm{Lie}(G_{\mathscr{E}})$ for some (any) field extension $\mathscr{E}$ of $\mathcal{F}$.

When the characteristic of $\mathcal{F}$ is *bad* for $G$, the adjoint orbits have some pathological properties – for example, in that case the scheme theoretic centralizer of a nilpotent element may fail to be smooth (i.e. reduced) over $\mathcal{F}$. To avoid such problems, we suppose $G$ to be a *standard* reductive group. Among other nice properties, one knows when $G$ is standard reductive that the centralizer $C_G(X)$ of any $X \in \mathrm{Lie}(G)$ is smooth over $\mathcal{F}$; see section 2.1 for a summary of this and other properties of standard reductive groups.

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[1] Here we view the $\mathcal{F}$-vector space $\mathrm{Lie}(G)$ as "being" an affine $\mathcal{F}$-variety; one might write $X \in \mathrm{Lie}(G)(\mathcal{F})$ to emphasize that $X$ is a $\mathcal{F}$-rational elements. We'll mostly avoid this more cumbersome notation in the sequel, however.

*Definition* 1.1.1. Let $G$ be a standard reductive group over the field $\mathcal{F}$, and let $X \in \mathrm{Lie}(G)$ be a nilpotent element. We say that $X$ is *geometrically distinguished* provided that any $\mathcal{F}$-torus of $C_G(X)$ is central in $G$.

Since $C_G(X)$ is smooth over $\mathcal{F}$, $X$ is geometrically distinguished if and only if $X \in \mathrm{Lie}(G_{\mathcal{F}_{\mathrm{alg}}})$ is distinguished in the usual sense – e.g. as in [Car93, §5.7] – where $\mathcal{F}_{\mathrm{alg}}$ is an algebraic closure of $\mathcal{F}$; here and elsewhere, the notation $G_{\mathcal{F}_{\mathrm{alg}}}$ denotes the group obtained from $G$ by *base-change*.

In this paper, we are primarily interested in *local fields*. Throughout the paper, $\mathcal{A}$ will denote a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, with field of fractions $\mathcal{K}$, and with residue field $k = \mathcal{A}/\mathfrak{m}$. We consider $\mathcal{F} = \mathcal{K}$, and we write $p > 0$ for the characteristic of $k$.

We make no further assumptions on $\mathcal{A}$; the residue field $k$ is not required to be perfect, and the characteristic of $\mathcal{K}$ may be either 0 (the "mixed characteristic" case) or $p$ (the "equal characteristic" case).

The results of the paper remain valid for arbitrary $k$. However the main contributions of the present work are to the setting of positive residue characteristic, and we have not formulated all statements and arguments to cover the case in which $k$ has characteristic 0.

1.2. **Reductive groups and some subgroups.** If $G$ is a connected and reductive group over a field $\mathcal{F}$, a subgroup $M \subset G$ is said to be of type $C(\mu)$ if $M$ is the connected centralizer $C^0_G(\phi)$ of the image of a homomorphism $\phi : \mu_N \to G$ for some $N$, where $\mu_N$ is the finite group scheme of "N-th roots of unity". If $M$ is such a subgroup, then $M$ is reductive and contains a maximal torus of $G$. More details on such subgroups are give in Section 2.2.

Now suppose $\mathcal{F} = \mathcal{K}$ is a *local field*. The reductive group $G$ is said to be *unramified* if there is a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ for which $G = \mathcal{G}_{\mathcal{K}}$. We say that $\mathcal{G}$ is a *reductive model* of $G$ over $\mathcal{A}$.

An algebraic torus $T$ over $\mathcal{K}$ is of course a reductive group. We observe in Corollary 2.3.3 below that a torus $T$ is *unramified* if and only if $T_{\mathcal{L}}$ is split for some finite unramified extension $\mathcal{K} \subset \mathcal{L}$.

The reductive group $G$ *splits over an unramified extension* of $\mathcal{K}$ provided that $G$ has a maximal $\mathcal{K}$-torus that is unramified. If $G$ is unramified, then $G$ splits over an unramified extension of $\mathcal{K}$; see Proposition 2.3.7.

1.3. **The main result.** The main result of this paper locates a nilpotent element of $\mathrm{Lie}(G)$ in a reductive subgroup of $G$ with some favorable properties. In some sense, our result may be viewed as a "$\mathcal{K}$-rational analogue" of part of the Bala-Carter Theorem.

**Theorem 1.3.1.** *Suppose that* $p > 2h - 2$. *Let* $X_1 \in \mathrm{Lie}(G)$ *be a nilpotent element. Then there is a* $\mathcal{K}$ *subgroup* $M \subset G$ *with the following properties:*

*(a)* $M$ *is an reductive group of type* $C(\mu)$ *containing a maximal unramified* $\mathcal{K}$*-torus of* $G$,
*(b)* $M$ *is an unramified reductive group over* $\mathcal{K}$, *and*
*(c)* $X_1 \in \mathrm{Lie}(M) \subset \mathrm{Lie}(G)$ *is geometrically distinguished for the action of* $M$.

*Remark* 1.3.2. In fact, the proof we will give shows a bit more. Indeed, we will establish that for a suitable reductive model $\mathcal{M}$ of the unramified group $M$, the nilpotent element $X_1$ has the form $\mathcal{X}_{\mathcal{K}}$ for a certain *balanced nilpotent section* $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$; see Section 1.4 and Section 1.6 below.

The Bala-Carter classification of nilpotent orbits for a standard reductive group $H$ over an algebraically closed field $\mathcal{F}$ was proved in "good characteristic" by Pommerening [Pom77] [Pom80], with a later proof – independent of case checking – given by Premet [Pre03]. See also [Car93, §5.9] and [Jan04]. In part, the Bala-Carter Theorem shows that the nilpotent $H$-orbits in $\mathrm{Lie}(H)$ are in bijection with conjugacy classes of pairs $(L, \mathcal{O})$ where $L$ is a Levi factor of a parabolic subgroup of $H$ and $\mathcal{O}$ is a distinguished nilpotent $L$-orbit in $\mathrm{Lie}(L)$. For $X \in \mathrm{Lie}(H)$, one chooses a maximal torus $S$ of the centralizer $C_H(X)$ and takes $L = C_H(S)$; then $X \in \mathrm{Lie}(L)$ is distinguished, and the Bala-Carter bijection is given by the assignment

$$X \mapsto (L, \mathrm{Ad}(L)X).$$

Return now to the reductive group $G$ over $\mathcal{K}$, and keep the assumptions of Theorem 1.3.1. For a fixed nilpotent element $X$ we may of course choose a maximal $\mathcal{K}$-torus $S \subset C_G(X)$. Setting $H = C_G(S)$, we find that $X \in \mathrm{Lie}(H)$ is indeed geometrically distinguished for $H$. However, the reductive group $C_G(S)$ need not be unramified; see Section 6.3.

1.4. **Balanced sections.** We are going to achieve the proof of Theorem 1.3.1 by relating nilpotent elements of G with nilpotent sections of certain $\mathcal{A}$-group schemes whose generic fiber identifies with G. We begin with some definitions.

Let $\mathcal{H}$ be a group scheme which is smooth, affine and of finite type over $\mathcal{A}$. The Lie algebra $\mathrm{Lie}(\mathcal{H})$ is of course a finitely generated free $\mathcal{A}$-module (an "$\mathcal{A}$-lattice"). Let us fix an element $\mathcal{X} \in \mathrm{Lie}(\mathcal{H})$ (a "section" over $\mathcal{A}$ or more precisely over $\mathrm{Spec}(\mathcal{A})$).

Write $C = C_{\mathcal{H}}(\mathcal{X})$ for the centralizer subgroup scheme (see Section 3.1).

*Definition* 1.4.1. $\mathcal{X}$ is a *balanced section* if $C_k = C_{\mathcal{H}_k}(\mathcal{X}_k)$ is smooth over k, if $C_{\mathcal{K}} = C_{\mathcal{H}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ is smooth over $\mathcal{K}$, and if $\dim C_k = \dim C_{\mathcal{K}}$.

See Section 3.1 for more details on the notion of a *balanced section*.

The section $\mathcal{X}$ is *nilpotent* provided that $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(G)$ is nilpotent, in which case $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{H}_k)$ is also nilpotent; see Lemma 3.2.1

We are mainly interested in these notions when the generic fiber $\mathcal{H}_{\mathcal{K}} = G$ is reductive over $\mathcal{K}$.

1.5. **Parahoric group schemes and balanced sections.** Let $\mathcal{P}$ be a parahoric group scheme over $\mathcal{A}$ with generic fiber $\mathcal{P}_{\mathcal{K}} = G$. These group schemes were introduced by Bruhat-Tits [BT84] and are parameterized by the points of the affine building associated with G. If G is unramified then a reductive model $\mathcal{G}$ is an important example of a parahoric group scheme. In general, however, the special fiber $\mathcal{P}_k$ need not be reductive; thus, the parahoric group schemes $\mathcal{P}$ for G are in general not reductive over $\mathcal{A}$.

For our purposes here, we will use the following description. Fix a maximal split torus S of G and write $V = X^*(S) \otimes \mathbf{Q}$. Then each point $x \in V$ determines a parahoric group scheme $\mathcal{P}_x$, and up to $G(\mathcal{K})$-conjugacy any parahoric group scheme has this form; see e.g. [McN20, §4.3]. Each $\mathcal{P} = \mathcal{P}_x$ is a smooth affine group scheme over $\mathcal{A}$ with $\mathcal{P}_{\mathcal{K}} = G$, and the special fiber $\mathcal{P}_k$ is a *connected* linear algebraic group over k.

Recent work of the author yields the following result about $\mathcal{P}$:

**Theorem 1.5.1** ([McN20])**.** *Suppose that* G *splits over an unramified extension of* $\mathcal{K}$*. There is a subgroup scheme* $\mathcal{M} \subset \mathcal{P}$ *such that*

*(a)* $\mathcal{M}$ *is reductive over* $\mathcal{A}$*,*
*(b)* $\mathcal{M}_k$ *is a Levi factor of* $\mathcal{P}_k$*, and*
*(c)* $\mathcal{M}_{\mathcal{K}}$ *is a reductive subgroup of* $G = \mathcal{P}_{\mathcal{K}}$ *of type* $C(\mu)$ *containing a maximal, maximally split torus of* G*.*

We explain statement (b). First of all, [McN20, Prop. 4.3.7] shows that the unipotent radical $R = R_u\mathcal{P}_k$ is defined over k; thus $\mathcal{P}_k/R$ is a linear algebraic group over k. Now the statement that $\mathcal{M}_k$ is a *Levi factor* of $\mathcal{P}_k$ means that the quotient mapping $\pi : \mathcal{P}_k \to \mathcal{P}_k/R$ yields on restriction an isomorphism $\pi_{|M} : M \to \mathcal{P}_k/R$ where $M = \mathcal{M}_k$.

1.6. **Unramified reductive groups and balanced sections.** We now consider an unramified reductive group G over $\mathcal{K}$ with reductive $\mathcal{A}$-model $\mathcal{G}$. The reductive group scheme $\mathcal{G}$ is said to be *standard* provided that $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_k$ are standard. This condition is immediate e.g. if $p > h$; see Proposition 2.2.2.

**Theorem 1.6.1.** *Suppose that* $\mathcal{G}$ *is a* standard *reductive group scheme over* $\mathcal{A}$*.*

*(a) For each nilpotent* $X \in \mathrm{Lie}(\mathcal{G}_k)$*, there is a balanced nilpotent section* $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ *with* $\mathcal{X}_k = X$*.*
*(b) If* $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ *is a balanced nilpotent section, and if* $\mathcal{X}_k$ *is geometrically distinguished, then* $\mathcal{X}_{\mathcal{K}}$ *is geometrically distinguished.*
*(c) Suppose that* $p > 2h-2$*. If* $\mathcal{X}, \mathcal{X}' \in \mathrm{Lie}(\mathcal{G})$ *are balanced nilpotent sections for which* $\mathcal{X}_k = \mathcal{X}'_k$*, there is* $g \in \mathcal{G}(\mathcal{A})$ *for which* $\mathcal{X}' = \mathrm{Ad}(g)\mathcal{X}$*.*

When $p > 2h-2$, consider the assignment $X \mapsto \mathcal{X}_{\mathcal{K}}$, where $\mathcal{X}$ is a balanced nilpotent section with $\mathcal{X}_k = X$. The Theorem shows that this assignment determines a well-defined mapping from the rational nilpotent orbits of $\mathcal{G}_k$ to the rational nilpotent orbits of G, and it shows that this mapping takes geometrically distinguished orbits for $\mathcal{G}_k$ to geometrically distinguished orbits for G.

1.7. **Balanced sections for parahoric group schemes.** In order to establish Theorem 1.3.1, we require an analogue of Theorem 1.6.1 valid for non-reductive parahoric group schemes. To establish this analogue, we will use $SL_2$-homomorphisms; in fact, these homomorphisms already play a role in the proof of the conjugacy statement Theorem 1.6.1(c).

Let $H$ be a standard reductive group over a field $\mathcal{F}$, and let $X \in \mathrm{Lie}(H)$ be nilpotent. When $\mathcal{F}$ has positive characteristic, the $\mathfrak{sl}_2$-triples provided by the Jacobson-Morozov Lemma are not available in general; however, one can instead associate to $X$ an *optimal cocharacter* $\phi : \mathbf{G}_m \to H$ which is a useful replacement; see Section 3.3 for further details.

Suppose that $X^{[p]} = 0$. Identifying the diagonal torus $\mathscr{D}$ of $SL_2$ with $\mathbf{G}_m$, we showed in [McN05] that the choice of $\phi$ determines a unique homomorphism $\Phi : SL_{2,\mathcal{F}} \to H$ for which $d\Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$ and $\Phi_{|\mathscr{D}} = \phi$. Such $\Phi$ are the *optimal* $SL(2)$*-homomorphisms*; see Section 4.

In this paper, we prove that the construction of optimal SL(2)-homomorphisms can be carried out over $\mathcal{A}$. We establish the following:

**Theorem 1.7.1.** *Let $\mathcal{G}$ be a* standard *reductive group scheme over $\mathcal{A}$ and let $X \in \mathrm{Lie}(\mathcal{G}_k)$ be a nilpotent element for which $X^{[p]} = 0$. Then there is a homomorphism of $\mathcal{A}$-group schemes $\Phi : SL_{2,\mathcal{A}} \to \mathcal{G}$ such that*
*(a) $\mathcal{X} = d\Phi(E_{\mathcal{A}})$ is a balanced nilpotent section with $\mathcal{X}_k = X$,*
*(b) $\Phi_k$ is an optimal homomorphism for $\mathcal{X}_k$, and*
*(c) $\Phi_{\mathcal{K}}$ is an optimal homomorphism for $\mathcal{X}_{\mathcal{K}}$.*

Now consider a reductive group $G$ over $\mathcal{K}$ which splits over an unramified extension, and let $\mathcal{P}$ be any parahoric $\mathcal{A}$-group scheme with $\mathcal{P}_{\mathcal{K}} = G$. Let us choose a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in Theorem 1.5.1. Of course, $\mathrm{Lie}(\mathcal{M}_k)$ identifies with $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$. Our main application of Theorem 1.7.1 is to achieve the proof of the following:

**Theorem 1.7.2.** *Suppose that $p > 2h-2$, and let $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$ be a nilpotent element.*
*(a) According to Theorem 1.6.1, there is a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ such that $\mathcal{X}_k = X_0$ and such that $\mathcal{X}$ is balanced for the action of $\mathcal{M}$. For any such section, $\mathcal{X} \in \mathrm{Lie}(\mathcal{M}) \subset \mathrm{Lie}(\mathcal{P})$ is balanced for the action of $\mathcal{P}$ as well.*
*(b) If $\mathcal{X}, \mathcal{X}' \in \mathrm{Lie}(\mathcal{P})$ are balanced nilpotent sections with ${\mathcal{X}'}_k = \mathcal{X}_k = X_0$ then $\mathcal{X}' = \mathrm{Ad}(g)\mathcal{X}$ for $g \in \mathcal{P}(\mathcal{A}) \subset G(\mathcal{K})$.*

The Theorem shows that – under its assumptions and notations – the assignment $X \mapsto \mathcal{X}_{\mathcal{K}}$ determines a well-defined mapping from the rational nilpotent orbits of $\mathcal{M}_k \subset \mathcal{P}_k$ to the rational nilpotent orbits of $G$, where $\mathcal{X}$ is a balanced nilpotent section with $\mathcal{X}_k = X$. In order to prove Theorem 1.3.1, we require to know that as we vary $\mathcal{P}$, the images of this assignment "account for" every rational nilpotent orbit for $G$. This is indeed the case:

**Theorem 1.7.3.** *Suppose that $p > 2h-2$, and let $X_1 \in \mathrm{Lie}(G)$ be nilpotent. Then there is a parahoric $\mathcal{A}$-group scheme $\mathcal{P}$ with $G = \mathcal{P}_{\mathcal{K}}$, a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ with the properties described in Theorem 1.5.1, and a balanced nilpotent section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ with $X_1 = \mathcal{X}_{\mathcal{K}}$.*

1.8. **An overview of the paper.** After some generalities about reductive groups and reductive group schemes in Section 2, we begin the discussion of balanced sections in Section 3. After some preliminary results about nilpotent orbits in standard reductive groups found in Section 3.3 – we give the proof of part (a) and (b) of Theorem 1.6.1 in Section 3.4.

Results concerning $SL_2$-homomorphisms are found in Section 4. In particular, we give the proof of Theorem 1.7.1 in Section 4.3; see Remark 4.3.8.

The existence of optimal $SL_2$-homomorphisms over $\mathcal{A}$ plays an important role in our proofs for existence for non-reductive parahoric group schemes; see Proposition 5.1.2. Using these methods, the proof of Theorem 1.7.2(a) – the existence result for balanced sections – is given in Section 5.1. And the conjugacy results Theorem 1.6.1(c) and Theorem 1.7.2(b) are proved in Section 5.2.

Finally, the proofs of Theorem 1.7.3 and of the main result Theorem 1.3.1 are given in Section 6.1.

Let $G$ be a reductive group over $\mathcal{K}$. In the *depth zero case* of the result of [DeB02], DeBacker established – under the assumption that the residue characteristic $p$ is zero or "sufficiently large" – a parametrization of the rational nilpotent orbits of $G$. His result labels the rational nilpotent orbits of $G$ using the (geometrically) distinguished rational nilpotent orbits of the reductive quotients $\mathcal{P}_k/R_u\mathcal{P}_k$ for the various parahoric group schemes $\mathcal{P}$ having generic fiber $\mathcal{P}_{\mathcal{K}} = G$.

In Section 6.2 we relate DeBacker's description with the balanced sections Theorem 1.7.2 described in this paper.

We provide examples which call attention to important aspects of the results in Section 6.3.

Finally, in an appendix Appendix A we give a proof in a more general context of a result from [MS03] about *cocharacters* associated to a nilpotent element and reductive subgroups of type $C(\mu)$.

1.9. **Remarks and notations.** Recall [Jan03, (I.2.7)] that if $\Lambda$ is a commutative ring and if $\mathcal{H}$ is an affine $\Lambda$-group scheme, then a module - or representation - for $\mathcal{H}$ is a $\Lambda$-module $M$ together with an action of $\mathcal{H}$ on the $\Lambda$-functor determined by $M$. In particular, for any commutative $\Lambda$-algebra $A$, the group $\mathcal{H}(A)$ acts linearly on $M(A) = M \otimes_\Lambda A$. Alternatively, one may view $M$ as a *comodule* for the Hopf algebra $\Lambda[\mathcal{H}]$.

If $\Lambda$ is a commutative ring, and if $D$ is a diagonalizable group scheme over $\Lambda$ with character group $X$, recall that any $D$-module $M$ can be written as a direct sum $M = \bigoplus_{\lambda \in X} M_\lambda$ where $D$ acts on the *weight space* $M_\lambda$ according to the character $\lambda$. If $\mathcal{H}$ is a $\Lambda$-group scheme, if $\phi : \mathbf{G}_m \to \mathcal{H}$ is a $\Lambda$-homomorphism, and if $M$ is an $\mathcal{H}$-module, we identify the character group of $\mathbf{G}_m$ with $\mathbf{Z}$, and for $i \in \mathbf{Z}$ we often write $M(\phi; i)$ for the $i$-weight space for the action of $\mathbf{G}_m$ on $M$ determined by $\phi$.

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1.11. **Data sharing statement.** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

## 2. Reductive groups and related group schemes

In this section, we collect some results on reductive groups and reductive group schemes.

2.1. **Standard reductive groups.** We describe the notion of "standard" reductive groups; the terminology follows the "standard hypotheses" considered by J. C. Jantzen - see e.g. [Jan04].

The class $\mathcal{C} = \mathcal{C}_\mathcal{F}$ of *standard reductive groups* over $\mathcal{F}$ consists of all connected and reductive linear algebraic groups over $\mathcal{F}$ satisfying the following properties:

(S1) $\mathcal{C}$ contains all simple $\mathcal{F}$-groups in *very good* characteristic; see Remark 2.1.1 below.
(S2) If $G_1$ and $G_2$ are in $\mathcal{C}$ then $G_1 \times G_2$ is in $\mathcal{C}$.
(S3) If $G$ is in $\mathcal{C}$ and $H$ is a reductive $F$-group, and if there is a separable isogeny between $G$ and $H$, then $H$ is in $\mathcal{C}$.
(S4) If $G$ is in $\mathcal{C}$ and $D \subset G$ is a diagonalizable subgroup scheme, then $C_G(D)^o$ is in $\mathcal{C}$.
(S5) If $G \simeq H \times T$ for a reductive $\mathcal{F}$-group $H$ and a $\mathcal{F}$-torus $T$, then $G$ is in $\mathcal{C}$ if and only if $H$ is in $\mathcal{C}$.

In this section, we are going to recall results concerning the nilpotent orbits of a geometrically standard reductive group over a ground field $\mathcal{F}$ of characteristic $p \geqslant 0$.

*Remark* 2.1.1. Recall if $H$ is a split simple group over $F$, the characteristic $p$ of $F$ is *good* for $H$ provided that $p$ does not divide the index of the root lattice in the weight lattice; see [SS70].

Suppose $p$ is good for the split group $H$ and that $H$ has root system $R$. Then $p$ is said to be *very good* for $H$ provided that $R$ has no irreducible component of type $A_r$ with the property $r \equiv -1 \pmod{p}$.

If $H$ is any split semisimple group, one may apply [Knu+98, Theorems 26.7 and 26.8] to see that there is a (possibly inseparable) central isogeny

$$\prod_{i=1}^{m} H_i \to H$$

where each $H_i$ is a simple group over $F$. The characteristic is *good* - respectively *very good* – for $H$ just in case it is good - respectively very good - for each $H_i$.

*Remark* 2.1.2. See [MT16, §4] for some discussion reconciling this version of the definition of "standard" with that found in older papers of the author. This discussion also compares these definitions with the notion of "pretty good primes" introduced by S. Herpel [Her13] and with the "standard hypotheses" of J. C. Jantzen.

*Remark* 2.1.3. For any $n \geqslant 1$, the group $GL_n$ is standard. The group $SL_n$ is D-standard if and only if $p$ does not divide $n$. See [McN05, Remark 3].

We say that $G$ is *geometrically standard* if $G_{\mathcal{E}}$ is standard for some finite and separable field extension $\mathcal{E}$ of $\mathcal{F}$.

**Proposition 2.1.4.** *Let $G$ be a geometrically standard reductive group over the field $\mathcal{F}$.*

*(a) The (scheme theoretic) center $Z = Z(G)$ of $G$ is geometrically smooth over $\mathcal{F}$.*
*(b) There is a $G$-invariant non-degenerate bilinear form $\beta$ on $\mathrm{Lie}(G)$, and $\mathrm{Lie}(G)$ is a completely reducible $G$-module.*
*(c) Let $G$ be a standard reductive group over $\mathcal{F}$, let $x \in G(\mathcal{F})$ and let $X \in \mathfrak{g}$. Then $C_G(x)$ and $C_G(X)$ are smooth over $\mathcal{F}$. In other words,*

$$\dim C_G(x) = \dim \mathfrak{c}_{\mathfrak{g}}(x) \quad \textit{and} \quad \dim C_G(X) = \dim \mathfrak{c}_{\mathfrak{g}}(X).$$

*Proof.* For standard reductive groups, (a) follows from [MT09, (3.4.2)], and (b) and (c) follow from [MT07, Prop. 12].

Now (a) and (c) follow at once for geometrically standard groups. For (b), also completely reducibility is a geometric property. As to the remaining assertion, recall that a non-degenerate bilinear form on $\mathrm{Lie}(G)$ amounts to a $G$-module isomorphism $\mathrm{Lie}(G) \to \mathrm{Lie}(G)^\vee$. Such an isomorphism exists over $\mathcal{F}$ if $G$ is standard. If $\mathcal{F}$ is infinite, the $\mathcal{F}$-homomorphisms $\mathrm{Hom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ form a dense subset of the variety of all $G$-homomorphisms, and it follows that the non-empty, open subvariety $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ must have an $\mathcal{F}$-rational point. If $\mathcal{F}$ is finite, note that $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ is a torsor over the group $A = \mathrm{Aut}_G(\mathrm{Lie}(G))$. Since $\mathrm{Lie}(G)$ is a semisimple $G$-module, $A$ is a connected (and reductive) $\mathcal{F}$-group. Hence the Lang-Steinberg Theorem implies that $\mathrm{Isom}_G(\mathrm{Lie}(G), \mathrm{Lie}(G)^\vee)$ has an $\mathcal{F}$-rational point, as required; see e.g. [Ser02, III §2]. □

**Proposition 2.1.5.** *Let $G$ be a reductive group over $\mathcal{F}$, let $h$ be the Coxeter number of $G$, and let $p$ denote the characteristic of $\mathcal{F}$. If $p = 0$ or if $p > h$, then $G$ is a standard reductive group.*

*Proof.* Let $G_1$ denote the derived group of $G$, let $Z$ denote the identity component of the center of $G$, and consider the product mapping

$$\Phi : Z \times G_1 \to G.$$

Then $\Phi$ is an isogeny [Spr98, Cor. 8.1.6 and Prop. 8.1.8(i)]. If the characteristic is zero, then $\Phi$ is automatically separable, hence $G$ is standard by (S3) and (S5). Now suppose that $p > 0$. Since $p > h$, it is immediate that $p$ is *very good* for $G_1$. Now, the kernel of $\Phi$ is isomorphic to the center $\zeta$ of $G_1$; since the characteristic is very good for $G_1$, it follows e.g. from [MT16, Remark 4.4(iii)] that $\zeta$ is a smooth group scheme. Thus indeed $\Phi$ is a separable isogeny so that $G$ is standard – again by (S3) and (S5) – as required. □

A reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ is *standard* provided that the fiber $\mathcal{G}_k$ is a geometrically standard reductive group over $k$, and the fiber $\mathcal{G}_{\mathcal{K}}$ is a geometrically standard reductive group over $\mathcal{K}$.

Recall that a reductive group $G$ over $\mathcal{K}$ is said to be *unramified* if there is a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ for which $G = \mathcal{G}_{\mathcal{K}}$.

**Proposition 2.1.6.** *Let $G$ be an unramified reductive group over $\mathcal{K}$, and suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are reductive models of $G$ over $\mathcal{A}$. Then $\mathcal{G}_1$ is geometrically standard if and only if $\mathcal{G}_2$ is geometrically standard.*

*Proof.* According to [$\mathrm{SGA3}_{III}$, p. XXII.2.8], the *type* of a reductive $\mathcal{A}$-group is locally constant on the spectrum of $\mathcal{A}$. It then follows from [$\mathrm{SGA3}_{III}$, XXIII Cor. 5.3] that $G_{1,\overline{k}}$ is isomorphic to $G_{2,\overline{k}}$; the result now follows. □

2.2. **Subgroups of a reductive group of type $C(\mu)$.** Let $G$ be a reductive group over an arbitrary field $\mathcal{F}$. In the introduction Section 1.2 we introduced the notion of a reductive subgroup of type $C(\mu)$.

This class of groups was studied in [McN20] and is slightly larger than the class of so-called *pseudo-Levi subgroups* considered in [MS03].

We denote by $\mu_n = \mu_{n,\mathcal{F}}$ the group scheme of $n$-th roots of unity in $\mathcal{F}$; it is the affine scheme with coordinate algebra $\mathcal{F}[T]/\langle T^n - 1\rangle$. The group scheme $\mu_n$ is *diagonalizable* over $\mathcal{F}$, and the group of characters $X^*(\mu_n)$ identifies with $\mathbf{Z}/n\mathbf{Z}$.

By a $\mu$-homomorphism with values in $G$, we mean an equivalence class of homomorphisms of group schemes $\phi : \mu_n \to G$ for varying $n$, under a natural equivalence relation – see [McN20, §3]

Following the terminology of [McN20], we say that a connected subgroup $M$ of $G$ is of type $C(\mu)$ if $M$ is the identity component of the centralizer in $G$ of the image of a homomorphism $\phi : \mu_n \to G$ [2]. Any subgroup of type $C(\mu)$ is reductive and contains a maximal torus of $G$; see [McN20, Prop. 3.4.1 and Theorem 3.4.6].

When $G$ is $\mathcal{F}$-split, [McN20, Theorem 3.4.5] gives an explicit description of the subgroups of type $C(\mu)$ containing a fixed maximal split torus.

**Proposition 2.2.1.** *Let $M$ be reductive subgroup of $G$ of type $C(\mu)$, let $h_G$ be the Coxeter number of $G$, and let $h_M$ be the corresponding value for $M$. Then $h_G \geqslant h_M$.*

*Proof.* It suffices to give the proof after scalar extension; thus, we may suppose that $G$ and $M$ are split. Moreover, the Coxeter numbers depend only on the root systems $R' = R_M$ and $R = R_G$ of $M$ and $G$.

We may construct the complex semisimple Lie algebras $\mathfrak{g}_{R'}, \mathfrak{g}_R$ with the indicated root systems; According to [McN20], $R' = R_M$ identifies with a closed symmetric subroot system of $R = R_G$. Thus we may identify $\mathfrak{g}_{R'}$ with a subalgebra of $\mathfrak{g}_R$.

Now, one knows that $h_G$ and $h_M$ are determined as follows:

$$2h_G - 2 = \min\{i \in \mathbf{Z}_{>0} \mid 0 = \operatorname{ad}(X)^i \in \operatorname{End}(\mathfrak{g}_R) \quad \text{for all nilpotent } X \in \mathfrak{g}_R\}$$

and

$$2h_M - 2 = \min\{i \in \mathbf{Z}_{>0} \mid 0 = \operatorname{ad}(Y)^i \in \operatorname{End}(\mathfrak{g}_{R'}) \quad \text{for all nilpotent } Y \in \mathfrak{g}_{R'}\}.$$

For nilpotent $Y \in \mathfrak{g}_{R'} \subset \mathfrak{g}_R$, it follows that

$$0 = \operatorname{ad}(Y)^{2h_G - 2} \in \operatorname{End}(\mathfrak{g}_R) \implies 0 = \operatorname{ad}(Y)^{2h_G - 2} \in \operatorname{End}(\mathfrak{g}_{R'})$$

and one concludes $2h_M - 2 \leqslant 2h_G - 2$. □

**Proposition 2.2.2.** *Let $G$ be a reductive group over $\mathcal{K}$, assume that $G$ splits over an unramified extension of $\mathcal{K}$, and let $\mathcal{P}$ be a parahoric group scheme with $G = \mathcal{P}_{\mathcal{K}}$.*

*(a) The unipotent radical of $\mathcal{P}_k$ is defined over $k$.*
*(b) If $p > h$ then $G_{\mathcal{K}}$ and $\mathcal{P}_k / R_u \mathcal{P}_k$ are standard reductive groups.*

*Proof.* Assertion (a) is confirmed in [McN20, Prop. 4.3.7 and 4.5.5]. For (b), it follows from Theorem 1.5.1 that there is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that $\mathcal{M}_k \simeq \mathcal{P}_k / R_u \mathcal{P}_k$ and such that $\mathcal{M}_{\mathcal{K}}$ is a reductive subgroup of $G$ of type $C(\mu)$.

Now, the root system of $\mathcal{M}_{\overline{k}}$ identifies with that of $\mathcal{M}_{\overline{\mathcal{K}}}$. If $h_M$ denotes the maximum Coxeter number of $M$, it follows from Proposition 2.2.1 that $h_M \leqslant h < p$. Now Proposition 2.1.5 shows that $\mathcal{M}_k$ is standard, as required. □

2.3. **Group schemes of multiplicative type, maximal tori, and cocharacters.** Let $\mathcal{G}$ and $\mathcal{M}$ be groups schemes of finite type over $\mathcal{A}$. Suppose moreover that $\mathcal{G}$ is smooth and affine over $\mathcal{A}$, and that $\mathcal{M}$ is of multiplicative type over $\mathcal{A}$. Consider the functors

$$F : \mathbf{Sch}_{\mathcal{A}} \to \mathbf{Sets} \quad \text{and} \quad H = \underline{\operatorname{Hom}}_{\mathcal{A}\text{-gr}}(\mathcal{M}, \mathcal{G}) : \mathbf{Sch}_{\mathcal{A}} \to \mathbf{Sets}$$

where $\mathbf{Sch}_{\mathcal{A}}$ is the category of schemes over $\mathcal{A}$, **Sets** is the category of sets, and for an $\mathcal{A}$-scheme $T$, $F(T)$ is the set of subgroup schemes of $\mathcal{G}$ of multiplicative type over $T$ and $H(T) = \operatorname{Hom}_{T\text{-gr}}(\mathcal{M}_T, \mathcal{G}_T)$ is the set of homomorphisms of group schemes $\mathcal{M}_T \to \mathcal{G}_T$ over $T$.

We record the following result:

**Theorem 2.3.1.** *(a) The functors $F$ and $H = \underline{\operatorname{Hom}}_{\mathcal{A}\text{-gr}}(\mathcal{M}, \mathcal{G})$ are represented by $\mathcal{A}$-schemes which are smooth and separated over $\mathcal{A}$.*
*(b) If $S_0 \subset \mathcal{G}_k$ is a $k$-torus, there is an $\mathcal{A}$-torus $S \subset \mathcal{G}$ for which $S_k = S_0$.*
*(c) If $\lambda_0 : \mathbf{G}_m \to \mathcal{G}_k$ and $\mu_0 : \mathcal{M}_k \to \mathbf{G}_m$ are $k$-homomorphisms, there is an $\mathcal{A}$-torus $\mathscr{S} \subset \mathcal{G}$ together with $\mathcal{A}$-homomorphisms $\lambda : \mathbf{G}_m \to \mathscr{T}$ and $\mu : \mathcal{M} \to \mathbf{G}_m$ for which $\mu_k = \mu_0$ and $\lambda_k : \mathbf{G}_{m,k} \to \mathscr{T}_k \subset \mathcal{G}_k$ coincides with $\lambda_0$.*

[2]We will permit ourselves to write $C_G(\phi)$ for the centralizer of the image of $\phi$, and $C_G^o(\phi)$ for the identity component of this centralizer.

*Proof.* The assertions in (a) are proved in [$SGA3_{II}$, Exp. XI Thm 4.1 and Cor. 4.2]. Now (b) and (c) are consequences of (a) together with the scheme-theoretic extension of Hensel's Lemma – see [$SGA3_{II}$, Exp. XI Prop 1.10 and Cor. 1.11]. □

**Corollary 2.3.2.** *Let $\mathscr{S}$ be a torus over $\mathcal{A}$. Then there is a DVR $\mathcal{B}$ which is a finite unramified extension of $\mathcal{A}$ for which $\mathscr{S}_{\mathcal{B}}$ is a split torus over $\mathcal{B}$.*

*Proof.* Write $S = \mathscr{S}_k$ for the special fiber of the torus $\mathscr{S}$, and choose a finite, separable extension $k \subset \ell$ for which the torus $S_\ell$ is split. Let $\mathcal{A} \subset \mathcal{B}$ be the corresponding unramified extension of $\mathcal{A}$; see e.g. [Ser79, III.6 Theorem 2]. Since $S_\ell$ is split, the rank of the character group $X^*(S_\ell)$ coincides with the dimension of $S$. Now Theorem 2.3.1(c) shows that the characters of $S_\ell$ prolong to $\mathcal{B}$-morphisms $\mathscr{S}_{\mathcal{B}} \to \mathbf{G}_m$, and one immediately deduces that the torus $\mathscr{S}_{\mathcal{B}}$ is indeed split over $\mathcal{B}$ as required.

□

**Corollary 2.3.3.** *Let $T$ be a $\mathcal{K}$ torus. Then $T$ has the form $T = \mathscr{T}_{\mathcal{K}}$ for some torus $\mathscr{T}$ over $\mathcal{A}$ if and only if $T$ splits over an unramified extension of $\mathcal{K}$.*

*Proof.* If $T$ splits over an unramified extension $\mathcal{L}$ of $\mathcal{K}$, we may and will suppose that $\mathcal{L}$is *Galois* over $\mathcal{K}$ with Galois group $\Gamma = \mathrm{Gal}(\mathcal{L}/\mathcal{K})$. If $\mathcal{B}$ denotes the ring of integers of $\mathcal{L}$, there is a canonical $\mathcal{B}$-torus $\mathscr{S}$ attached to the split $\mathcal{L}$-torus $T_{\mathcal{L}}$; see [BT84, §1.2.11]. Now $\Gamma$ acts on $\mathcal{B}[\mathscr{S}]$ and [BT84, §5.1.8] yields the existence of the required $\mathcal{A}$-torus $\mathscr{T}$.

On the other hand, if $T = \mathscr{T}_{\mathcal{K}}$ for some $\mathcal{A}$-torus $\mathscr{T}$, it follows from Corollary 2.3.2 that $\mathscr{T}$ - and hence also $T$ - splits over an unramified extension, as required. □

For a group scheme $\mathcal{H}$ which is smooth and of finite type over $\mathcal{A}$, a closed subgroup scheme $\mathscr{T} \subset \mathcal{H}$ is called a *maximal $\mathcal{A}$-torus* if: (i) $\mathscr{T}$ is an $\mathcal{A}$-torus, (ii) $\mathscr{T}_k$ is a maximal torus of $\mathcal{H}_k$ and (iii) $\mathscr{T}_{\mathcal{K}}$ is a maximal torus of $\mathcal{H}_{\mathcal{K}}$.

**Corollary 2.3.4.** *Let $\mathcal{G}$ be reductive over $\mathcal{A}$.*

*(a) If $S$ is a maximal $k$-torus of $\mathcal{G}_k$, there is a maximal $\mathcal{A}$-torus $\mathscr{S}$ of $\mathcal{G}$ with $\mathscr{S}_k = S$.*
*(b) In particular, $\mathcal{G}$ possesses a maximal torus $\mathscr{S}$ over $\mathcal{A}$.*
*(c) If $\phi : \mathbf{G}_{m,k} \to \mathcal{G}_k$ is a $k$-homomorphism, there is a maximal torus $\mathscr{S}$ of $\mathcal{G}$ and a homomorphism $\Phi : \mathbf{G}_m \to \mathscr{S}$ such that $\Phi_k : \mathbf{G}_{m,k} \to \mathscr{S}_k \to \mathcal{G}_k$ coincides with $\phi$.*

*Proof.* For (a), use Theorem 2.3.1 to find a torus $\mathscr{S} \subset \mathcal{G}$ with $\mathscr{S}_k = S$. We now argue that $\mathscr{S}$ is a maximal torus – i.e. that $\mathscr{S}_{\mathcal{K}}$ is a maximal torus of $\mathcal{G}_{\mathcal{K}}$. Since $\mathcal{G}$ is reductive, it follows from [$SGA3_{II}$, Exp XII, Théorèm 1.7] that $\mathcal{G}$ has a maximal torus locally in the étale topology of $\mathrm{Spec}(\mathcal{A})$. In particular, the dimension of a maximal torus of $\mathcal{G}_{\mathcal{K}}$ coincides with that of a maximal torus of $\mathcal{G}_k$. Since $\dim \mathscr{S}_{\mathcal{K}} = \dim \mathscr{S}_k$, it follows that $\mathscr{S}_{\mathcal{K}}$ is a maximal torus of $\mathcal{G}_{\mathcal{K}}$, as required; this proves (a).

Now (b) follows from (a). For (c), choose a maximal torus $S$ of $\mathcal{G}_k$ containing the image of $\phi$ and apply (a) together with Theorem 2.3.1. □

**Proposition 2.3.5.** *Suppose that $\lambda_0 : \mathbf{G}_m \to \mathrm{der}(\mathcal{G}_k)$ is a $k$-homomorphism with values in the* derived group *of $\mathcal{G}_k$. Then there is an $\mathcal{A}$-homomorphism $\lambda : \mathbf{G}_m \to \mathcal{G}$ such that $\lambda_k = \lambda_0$ and such that $\lambda_{\mathcal{K}}$ takes values in the derived group* $\mathrm{der}(\mathcal{G}_{\mathcal{K}})$.

*Proof.* According to [$SGA3_{III}$, Théorème 6.2.1], there is a closed subgroup scheme $\mathrm{der}(\mathcal{G})$ which is semisimple over $\mathcal{A}$ for which $\mathrm{der}(\mathcal{G})_k$ is the derived group of $\mathcal{G}_k$ and $\mathrm{der}(\mathcal{G})_{\mathcal{K}}$ is the derived group of $\mathcal{G}_{\mathcal{K}}$. Now the result follows from Theorem 2.3.1. □

The following terminology was given in the introduction:

*Definition* 2.3.6. A reductive algebraic group $G$ over $\mathcal{K}$ is said to be *unramified* if there is a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ for which $G = \mathcal{G}_{\mathcal{K}}$.

**Proposition 2.3.7.** *Let $G$ be an unramified reductive group over $\mathcal{K}$. Let $\mathcal{G}$ be a reductive model of $G$ over $\mathcal{A}$, and let $\mathscr{T}$ be a maximal $\mathcal{A}$-torus of $\mathcal{G}$; cf. Corollary 2.3.4.*

*(1) There is DVR $\mathcal{B}$ which is a finite unramified extension of $\mathcal{A}$ for which $\mathscr{T}_{\mathcal{B}}$ is a split torus over $\mathcal{B}$.*
*(2) In particular, $\mathcal{G}_{\mathcal{B}}$ is a split reductive group scheme over $\mathcal{B}$, and*
*(3) $G$ splits over an unramified extension of $\mathcal{K}$.*

*Proof.* (a) this follows from Corollary 2.3.2. Now (b) follows from [SGA3$_{III}$, Exp. XXII Prop. 2.2], and (c) is an immediate consequence. ☐

*Remark* 2.3.8. Of course, there are are reductive groups G over $\mathcal{K}$ which split over an unramified extension but which are not themselves unramified. For an example, suppose the residue field $k$ to be finite, let $Q$ be a non-split quaternion algebra over $\mathcal{K}$, and let $G = GL_{1,Q}$. It is well-known that $Q$ has an unramified splitting field $\mathcal{L}$, hence $G_{\mathcal{L}} \simeq GL_{2,\mathcal{L}}$ is split. On the other hand, G is anisotropic modulo its center. Now [BT84, Cor. 5.1.27] implies that there is an essentially unique parahoric group scheme – an Iwahori group scheme – with generic fiber G. Since I is not reductive, G has no reductive model.

**Proposition 2.3.9.** *Let* $H$ *be a (smooth) linear algebraic group over the local field* $\mathcal{K}$*. Then* $H$ *has a* $\mathcal{K}$*-torus* $T$ *which is maximal among unramified* $\mathcal{K}$*-tori contained in* $H$*.*

*Proof.* Since $H$ is smooth, it has maximal tori defined over $\mathcal{K}$. Write $\mathcal{K}_{un}$ for the maximal unramified extension of $\mathcal{K}$ in some fixed algebraic closure. For any maximal $\mathcal{K}$-torus $S$, [Spr98, Prop. 13.2.4] show that there are unique $\mathcal{K}_{un}$-subtori $S_a, S_s$ such that $S_a$ is anisotropic over $\mathcal{K}_{un}$, $S_s$ is split over $\mathcal{K}_{un}$, $S = S_a S_s$ and $S_a \cap S_s$ is finite. Since the sub-tori $S_a$ and $S_s$ are *unique*, they are stable by the Galois group and hence are defined over $\mathcal{K}$. We have, $\dim S_a + \dim S_s = \dim S$.

As the maximal $\mathcal{K}$-torus $S \subset H$ varies, write $d(S)$ for the dimension of the maximal torus $S_s$. Now choose a $\mathcal{K}$-torus $S$ for which $d(S)$ is *maximal.* It is then clear that the $\mathcal{K}$-torus $T = S_s$ has the required properties. ☐

2.4. **Parabolic subgroup schemes of a reductive group scheme.** Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$ with connected fibers. We first recall - see [SGA3$_{III}$, Exp XXII Def 5.11.1] - that a subgroup scheme $\mathcal{H} \subset \mathcal{G}$ is said to be *of type (R)* provided that $\mathcal{H}$ is smooth over $\mathcal{A}$ with connected fibers, $\mathcal{H}_k$ contains a maximal torus of $\mathcal{G}_k$, and $\mathcal{H}_{\mathcal{K}}$ contains a maximal torus of $\mathcal{G}_{\mathcal{K}}$.

Now, according to [SGA3$_{III}$, Exp XXVI §1] a subgroup scheme $P \subset \mathcal{G}$ is a *parabolic subgroup scheme* if $P$ is smooth over $\mathcal{A}$, if $P_k \subset \mathcal{G}_k$ is a parabolic subgroup, and if $P_{\mathcal{K}} \subset \mathbf{G}_{\mathcal{K}}$ is a parabolic subgroup. In particular, a parabolic subgroup scheme of $\mathcal{G}$ is of type (R).

If $P$ is a parabolic subgroup scheme of $\mathcal{G}$, then according to [SGA3$_{III}$, Exp. XXVI Prop. 1.6] there is a closed, normal subgroup scheme $R_u P \subset P$ such that $R_u P$ is smooth over $\mathcal{A}$ with connected fibers, $(R_u P)_k$ is the unipotent radical of $P_k$, and $(R_u P)_{\mathcal{K}}$ is the unipotent radical of $P_{\mathcal{K}}$.

**Proposition 2.4.1.** *Let* $\mathscr{S} \subset \mathcal{G}$ *be an* $\mathcal{A}$*-torus. Then the centralizer* $\mathcal{M} = C_{\mathcal{G}}(\mathscr{S})$ *is a reductive* $\mathcal{A}$*-subgroup scheme of* $\mathcal{G}$ *with connected fibers. Moreover,* $\mathcal{M}$ *is a subgroup scheme of* $\mathcal{G}$ *of type (R).*

*Proof.* Indeed, according to [SGA3$_{II}$, Exp XI, Cor 5.3], $\mathcal{M}$ is a closed subgroup scheme of $\mathcal{G}$ which is smooth over $\mathcal{A}$. Now [SGA3$_{III}$, Exp XIX, §1.3] shows that $\mathcal{M}_k$ and $\mathcal{M}_{\mathcal{K}}$ are connected and reductive subgroups of $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ respectively, and the Proposition follows. ☐

**Lemma 2.4.2.** *Let* $\mathcal{L}$ *be a free* $\mathcal{A}$*-module of finite rank, suppose that* $\psi : \mathbf{G}_m \to GL(\mathcal{L})$ *is an* $\mathcal{A}$*-homomorphism, and let* $\mathcal{M} \subset \mathcal{L}$ *be an* $\mathcal{A}$*-submodule such that*

$$\mathcal{M}_{\mathcal{K}} = \sum_{i \geqslant m} \mathcal{L}_{\mathcal{K}}(\psi_{\mathcal{K}}; i) \quad \textit{and} \quad \mathcal{M}_k = \sum_{i \geqslant m} \mathcal{L}_k(\psi_k; i).$$

*for some integer* $m$*. Then* $\mathcal{M} = \sum_{i \geqslant m} \mathcal{L}(\psi; i)$*.*

*Proof.* Since $\mathcal{L} = \bigoplus_i \mathcal{L}(\psi; i)$, we have

$$(\flat) \quad \mathcal{M} \subset \mathcal{L} \cap \left( \sum_{i \geqslant m} \mathcal{L}_{\mathcal{K}}(\psi_{\mathcal{K}}; i) \right) = \sum_{i \geqslant m} \mathcal{L}(\psi; i).$$

Since $\mathcal{M}_k = \sum_{i \geqslant m} \mathcal{L}_k(\psi_k; i)$ it follows from the Nakayama Lemma that equality holds in $(\flat)$. ☐

For more on the "limit group scheme" described in the following result, the reader can consult [CGP15, §2.1].

**Proposition 2.4.3.** *Let* $\psi : \mathbf{G}_m \to \mathcal{G}$ *be an* $\mathcal{A}$*-homomorphism. There is a unique subgroup scheme* $P_{\mathcal{G}}(\psi) \subset \mathcal{G}$ *of type (R) with the property that*

$$(\sharp) \quad \operatorname{Lie} P_{\mathcal{G}}(\psi) = \sum_{i \geqslant 0} \operatorname{Lie}(\mathcal{G})(\psi; i).$$

*Moreover,*

*(a)* $P = P_{\mathcal{G}}(\psi)$ *is a parabolic subgroup scheme of* $\mathcal{G}$*,*
*(b)* $C_{\mathcal{G}}(\psi)$ *is a Levi factor of* $P$*, and*
*(c)* $\operatorname{Lie}(R_u P) = \sum_{i>0} \operatorname{Lie}(G)(\psi; i)$.

*Proof.* First recall [Spr98, Prop. 8.4.5 and Theorem 13.4.2] that if $G$ is a connected and reductive group over a field $\mathcal{F}$ and if $\phi : \mathbf{G}_m \to G$ is an $\mathcal{F}$-homomorphism, then there is a parabolic subgroup $P_G(\phi)$ for which

$$P_G(\phi)(\mathcal{F}_{\mathrm{sep}}) = \{g \in G(\mathcal{F}_{\mathrm{sep}}) \mid \lim_{t \to 0} \operatorname{Int}(\phi(t))g \text{ exists}\}.$$

Moreover, $\operatorname{Lie}(P_G(\phi)) = \sum_{i \geqslant 0} \operatorname{Lie}(G)(\phi; i)$, $\operatorname{Lie}(R_u P_G(\phi)) = \sum_{i>0} \operatorname{Lie}(G)(\phi; i)$, and $C_G(\phi)$ is a Levi factor of $P_G(\phi)$.

Let us write $P_{\mathcal{K}}$ for the parabolic subgroup $P_{\mathcal{G}_{\mathcal{K}}}(\psi_{\mathcal{K}})$ of $\mathcal{G}_{\mathcal{K}}$ determined by $\psi_{\mathcal{K}}$. Consider the functor $\underline{\mathrm{Par}}$ defined for an $\mathcal{A}$-scheme $S$ by the rule

$$\underline{\mathrm{Par}}(S) = \text{set of all } S\text{-parabolic subgroup schemes of } \mathcal{H}_S$$

It follows from [SGA3$_{\mathrm{III}}$, Exp. XXVI, Cor. 3.5] that $\underline{\mathrm{Par}}$ is represented by a scheme which is smooth, projective and of finite type over $\mathcal{A}$.

Since $\underline{\mathrm{Par}}$ is projective and since $\mathcal{A}$ is a discrete valuation ring, the $\mathcal{K}$-points of this $\mathcal{A}$-scheme coincide with its $\mathcal{A}$-points; see e.g. [Liu02, Thm 3.3.25]. Thus, the $\mathcal{K}$-point $P_{\mathcal{K}} \in \underline{\mathrm{Par}}(\mathcal{K})$ determines a unique $\mathcal{A}$-point $P \in \underline{\mathrm{Par}}(\mathcal{A})$. Since $\operatorname{Lie}(P)$ is an $\mathcal{A}$-lattice in $\operatorname{Lie}(P_{\mathcal{K}})$ and since $\operatorname{Lie}(P)$ is contained in $\operatorname{Lie}(\mathcal{G})$, it follows immediately from Lemma 2.4.2 that $(\sharp)$ holds.

Since $P$ is smooth, the discussion in [BT84, (I.2.6)] shows that it is equal to the schematic closure in $\mathcal{G}$ of its generic fiber $P_{\mathcal{K}}$. Similarly, $\mathcal{M} = C_{\mathcal{G}}(\psi)$ is the schematic closure of $M_{\mathcal{K}} = C_{\mathcal{G}_{\mathcal{K}}}(\psi_{\mathcal{K}})$. Since $M_{\mathcal{K}} \subset P_{\mathcal{K}}$, it follows that $\mathcal{M}$ is a closed subgroup scheme of $P$.

Now, $\mathcal{M}$ is a reductive subgroup scheme of $P$, and according to the discussion in the first paragraph of this proof, $\mathcal{M}_k$ is a Levi factor of $P_k$ and $\mathcal{M}_{\mathcal{K}}$ is a Levi factor of $P_{\mathcal{K}}$. Thus indeed $\mathcal{M}$ is a Levi factor of $P$ and (b) holds.

Finally, recall the smooth subgroup scheme $R_u P$ has the property that

$$\operatorname{Lie}((R_u P)_{\mathcal{K}}) = \sum_{i>0} \operatorname{Lie}(\mathcal{G}_{\mathcal{K}})(\psi_{\mathcal{K}}; i) \quad \text{and} \quad \operatorname{Lie}((R_u P)_k) = \sum_{i>0} \operatorname{Lie}(\mathcal{G}_k)(\psi_k; i).$$

It now follows from Lemma 2.4.2 that (c) holds. □

## 3. Balanced nilpotent sections for reductive group schemes

The main goal of this section is the proof of assertions (a) and (b) of Theorem 1.6.1; this proof is given in Section 3.4 after some preliminaries.

3.1. **Generalities for balanced sections and their centralizers.** Recall that we introduced the notion of a balanced section in the introduction Section 1.4. In this section, we formulate some general results about balanced sections and their stabilizers.

First we recall what is meant by the scheme-theoretic *identity component*. For a group scheme $\mathcal{G}$ of finite type over $\mathcal{A}$, the identity component is a certain A-subgroup functor; see [SGA3$_{\mathrm{I}}$, VI.B.3.1]. If $\mathcal{G}$ is moreover smooth over $\mathcal{A}$, then the *identity component* $\mathcal{G}^0$ of $\mathcal{G}$ is the union of the identity components $\mathcal{G}^0_k$ and $\mathcal{G}^0_{\mathcal{K}}$ – see [BT84, §1.2.12] or [SGA3$_{\mathrm{I}}$, VI$_{\mathrm{B}}$ Thm. 3.10]; it is an open subgroup scheme of $\mathcal{G}$ which is smooth over $\mathcal{A}$ and has connected fibers. In particular, the generic fiber $(\mathcal{G}^0)_{\mathcal{K}}$ is precisely the identity component of $\mathcal{G}_{\mathcal{K}}$.

Next, we recall the follow result of Raynaud:

**Proposition 3.1.1.** *Let* $\mathbf{G}$ *be a flat group scheme of finite type over* $\mathcal{A}$ *such that the generic fiber* $\mathcal{G}_{\mathcal{K}}$ *is affine. Then* $\mathcal{G}$ *is affine over* $\mathcal{A}$ *if and only if it is separated over* $\mathcal{A}$*.*

*Proof.* A proof of this result can be found in [PY06, Prop. 3.1]. □

**Corollary 3.1.2.** *Let $\mathcal{G}$ be a smooth group scheme which is affine over $\mathcal{A}$. Then the identity component $\mathcal{G}^0$ is affine over $\mathcal{A}$.*

*Proof.* Since $\mathcal{G}$ is affine over $\mathcal{A}$, it and its open subgroup $\mathcal{G}^0$ are separated over $\mathcal{A}$. Now $(\mathcal{G}^0)_{\mathcal{K}} = \mathcal{G}^0_{\mathcal{K}}$ is closed in $\mathcal{G}_{\mathcal{K}}$ and is thus affine. So the the Corollary follows from Proposition 3.1.1. □

I thank an anonymous referee for pointing out the following reference and result; this result is simpler and more efficient than a previous argument.

**Theorem 3.1.3.** *Let $\mathcal{G}$ be a group scheme of finite type over $\mathcal{A}$. Then the following are equivalent:*

*(a) the fibers $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_k$ are smooth of the same dimension, and*
*(b) the $\mathcal{A}$-functor $\mathcal{G}^0$ is representable by an open $\mathcal{A}$-subgroup scheme which is smooth over $\mathcal{A}$.*

*Proof.* The statement (b) $\implies$ (a) is immediate, and (b) $\implies$ (a) follows from [SGA3$_I$, VI.B.4.4] □

Suppose now that $\mathcal{G}$ is a group scheme over $\mathcal{A}$ which is smooth, affine and of finite type over $\mathcal{A}$. Consider an $\mathcal{G}$-module $\mathcal{L}$ which is a finitely generated free $\mathcal{A}$-module. For $x \in \mathcal{L}$, write $x_{\mathcal{K}}$ for the image of $x$ in $\mathcal{L}_{\mathcal{K}} = \mathcal{L} \otimes_{\mathcal{A}} K$, and write $x_k$ for the image of $x$ in $\mathcal{L}_k = \mathcal{L} \otimes_{\mathcal{A}} k = \mathcal{L}/\mathfrak{m}\mathcal{L}$.

The stabilizer $C = \mathrm{Stab}_{\mathcal{G}}(x)$ is a closed – and hence affine – subgroup scheme of $\mathcal{G}$; see [Jan03, p. I.2.6]. We recapitulate the following definition from Section 1.4.

*Definition* 3.1.4. The element $x \in \mathcal{L}$ will be said to be *balanced* for the action of $\mathcal{G}$ if the scheme-theoretic stabilizer $C = \mathrm{Stab}_{\mathcal{G}}(x)$ has the following properties: (i) $C_{\mathcal{K}}$ is a smooth group scheme over $\mathcal{K}$, (ii) $C_k$ is a smooth group scheme over $k$, and (iii) $\dim C_{\mathcal{K}} = \dim C_k$.

We have the following immediate consequence of Theorem 3.1.3:

**Corollary 3.1.5.** *Let $x \in \mathcal{L}$ be a balanced section, and let $C = \mathrm{Stab}_{\mathcal{G}}(x)$. Then there is a open $\mathcal{A}$-subgroup scheme $\mathcal{H} \subset C$ such that*

*(a) $\mathcal{H}$ is smooth, affine and of finite type over $\mathcal{A}$, and*
*(b) $\mathcal{H}_{\mathcal{K}} = (C_{\mathcal{K}})^0 = \mathrm{Stab}_{\mathcal{H}_{\mathcal{K}}}(x_{\mathcal{K}})^0$ and $\mathcal{M}_k = (C_k)^0 = \mathrm{Stab}_{\mathcal{H}_k}(x_k)^0$.*

*Remark* 3.1.6. Let $\mathcal{G}$ be an $\mathcal{A}$-group scheme satisfying the equivalent conditions of Theorem 3.1.3, and write $\mathcal{M}$ for the smooth and open $\mathcal{A}$-subgroup scheme representing the identity component of $\mathcal{G}$. It is clear that $\mathcal{M}$ is preserved by any $\mathcal{A}$-automorphism of $\mathcal{G}$.

3.2. **Nilpotent sections.** In this paper, our main interest is in nilpotent sections. In this section, we record some generalities. Let $\mathcal{G}$ be a smooth, affine group scheme of finite type over $\mathcal{A}$.

**Lemma 3.2.1.** *Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G})(\mathcal{A})$ be a section. If $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ is nilpotent, then also $\mathcal{X}_k \in \mathrm{Lie}(\mathcal{G}_k)$ is nilpotent.*

*Proof.* Recall that if $H$ is a linear algebraic group over the field $\mathcal{F}$, then an element $Y \in \mathrm{Lie}(H)$ is nilpotent if and only if $0 = Y^N \in \mathrm{End}_{\mathcal{F}}(V)$ for some sufficiently large $N > 0$ and some faithful linear representation $V$ of $H$.

Since $\mathcal{G}$ is affine and smooth - in particular, flat - over $\mathcal{A}$, it follows from [BT84, (1.4.5)] that $\mathcal{G}$ has a faithful linear representation on a free $\mathcal{A}$-module $M$ of finite rank. Then $M_k = M \otimes_{\mathcal{A}} k$ affords a faithful linear representation of $\mathcal{G}_k$, and $M_{\mathcal{K}} = M \otimes_{\mathcal{A}} \mathcal{K}$ affords a faithful linear representation of $\mathcal{G}_{\mathcal{K}}$.

Identifying $\mathcal{G}$ as a closed subgroup scheme of $GL(M)$, the nilpotence of $\mathcal{X}_{\mathcal{K}}$ implies that $0 = \mathcal{X}_{\mathcal{K}}^N \in \mathrm{End}_{\mathcal{K}}(M_{\mathcal{K}})$ for some $N > 0$. But then $0 = \mathcal{X}^N \in \mathrm{End}_{\mathcal{A}}(M)$, and hence $0 = \mathcal{X}_k^N \in \mathrm{End}_k(M_k)$ so that indeed $\mathcal{X}_k$ is nilpotent as well. □

*Remark* 3.2.2. Of course, the Lemma applies when $\mathcal{G}$ is reductive over $\mathcal{A}$, or more generally when $\mathcal{G} = \mathcal{P}$ is a parahoric group scheme with generic fiber $G$.

*Definition* 3.2.3. A section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ is said to be *nilpotent* just in case the generic fiber $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ is nilpotent.

Thus the Lemma shows that the value $\mathcal{X}_k$ at the closed point of $\mathrm{Spec}(\mathcal{A})$ of a nilpotent section $\mathcal{X}$ is nilpotent.

3.3. **Nilpotent orbits for a reductive group over a field.** We consider a ground field $\mathcal{F}$ and a geometrically standard reductive group G over $\mathcal{F}$. We now recall for later use some important facts about the nilpotent orbits of G. Write $\mathfrak{g} = \text{Lie}(G)$ for the Lie algebra of G, let $X \in \mathfrak{g}$ be a nilpotent element, and let $C = C_G(X)$ for the centralizer of X in G.

*Definition* 3.3.1. A cocharacter $\phi : \mathbf{G}_m \to G$ is *associated with* X provided that

(a) $X \in \mathfrak{g}(\phi; 2)$, and
(b) there is a maximal torus $S \subset C$ such that the image of $\phi$ lies in the derived group $\text{der}(M)$ where $M = C_G(S)$.

This definition is essentially that given by Jantzen in [Jan04, §4] when working over an algebraically closed field, and - in slightly different language - by Premet in [Pre03]. In the case of an algebraically closed field, the result below – Proposition 3.3.2 – essentially follows from work of Premet [Pre03] giving a modern proof of the Bala-Carter Theorem.

Write $N = N(X)$ for the stabilizer in G of the line [X] in the projective space $\mathbf{P}(\mathfrak{g})$.

**Proposition 3.3.2.** *(a) The subgroup* N *is smooth over* $\mathcal{F}$*.*

*(b) For each maximal torus* S *of* N*, there is a unique cocharacter* $\phi \in X_*(S)$ *associated to* X*. In particular, there is a cocharacter associated with* X*.*
*(c) If* $\phi$ *is a cocharacter associated with* X*, the image of* $\phi$ *centralizes some maximal* $\mathcal{F}$*-torus of the connected centralizer* $H = C_G(X)$*, and the centralizer* $M = C_H(\phi)$ *of the image of* $\phi$ *in* H *is a Levi factor of* H*.*
*(d) If* S *is a maximal* $\mathcal{F}$*-torus of* $C_G(X)$*, there is a cocharacter associated with* X *whose image centralizes* S*.*
*(e) The unipotent radical* $U = R_u(C)$ *is defined and split over* $\mathcal{F}$*, and*
*(f) Any two associated cocharacters for* X *are conjugate by a unique element of* $U(\mathcal{F})$*.*

*Proof.* We are going to give references for the statements; these references assume that G is a standard reductive group. Since here we only suppose that G is geometrically standard, let us fix a finite separable extension $\mathcal{F}'$ of $\mathcal{F}$ for which $G_{\mathcal{F}'}$ is standard.

For (a), the smoothness of $N_{\mathcal{F}'}$ follows from [McN04, Lemma 23]; and the smoothness of N follows at once.

Now (b), since N is smooth over $\mathcal{F}$ we may choose a maximal $\mathcal{F}$-torus S of N. Now, [MT09, Prop. 15] shows that $S_{F'}$ has a unique cocharacter $\phi$ associated with X. Then $\phi$ is an $\mathcal{F}$-cocharacter by Galois descent.

Let us now fix an $\mathcal{F}$-cocharacter $\phi$ associated with X. For (c), we may form the centralizer $M = C_H(\phi)$, an $\mathcal{F}$-subgroup of the connected centralizer $H = C^0_G(X)$. Now [McN04, Cor. 20] shows that $M_{\mathcal{F}'}$ is a Levi factor of $H_{\mathcal{F}'}$, it is immediate that M is a Levi factor of H.

(d) is obtained by applying (b) to the nilpotent element X in the Lie algebra of the geometrically standard reductive $\mathcal{F}$-group $C_G(S)$.

Statement (e) holds for $U_{\mathcal{F}'}$ by [McN05, Prop/Defn 21, (3)]; the assertion for U now follows e.g. by [Spr98, Theorem 14.3.8].

Finally, consider (f) and let $\phi_1, \phi_2$ be $\mathcal{F}$-cocharacters of G associated with X. We may apply [McN05, Prop/Defn 21, (4)] to $G_{\mathcal{F}'}$ to learn that $\phi_1$ and $\phi_2$ are conjugate by a unique element $u \in U(\mathcal{F}')$. But the unipotent radical U is an $\mathcal{F}'$-group, and the uniqueness of u permits us to conclude that $u \in U(\mathcal{F})$ as required. ☐

An cocharacter associated to X determines the dimension of the orbit of X, as follows:

**Proposition 3.3.3.** *Let* $\phi$ *be a cocharacter associated to* X*, let* $P(\phi)$ *be the parabolic subgroup determined by* $\phi$*, and write* $C_G(X)$ *for the centralizer of* X *in* G*. Then:*

*(a)* $C_G(X) \subset P(\phi)$*, and*
*(b)* $\dim C_G(X) = \dim_{\mathcal{F}}(\mathfrak{g}(\phi; 0) + \mathfrak{g}(\phi; 1))$*.*

*Proof.* Indeed, [Jan04, Prop. 5.9] shows that (a) holds, that the P-orbit $\mathcal{O}$ of X is smooth – where $P = P(\phi)$ – and that $\mathcal{O}$ is dense in

$$R = \sum_{i \geqslant 2} \mathfrak{g}(\phi; i).$$

The result now follows since $\text{Lie}(P(\phi)) = \sum_{i \geqslant 0} \mathfrak{g}(\phi; i)$. ☐

The Richardson orbits will play an important role.

**Proposition 3.3.4.** *Let $Q \subset G$ be an $\mathcal{F}$-parabolic subgroup. Then $Q$ has a unique open and separable orbit on $\mathrm{Lie}(R_Q)$ containing an $\mathcal{F}$-rational point.*

*Proof.* We first argue that when $G$ is $\mathcal{F}$-split, there is a unique open and separable $Q$-orbit on $\mathrm{Lie}(R_Q)$ defined over $\mathcal{F}$. When $\mathcal{F}$ is algebraically closed, this follows e.g. from [Jan04, §4.9]; the separability of this $Q$-orbit follows from [Jan04, §4.9](B) since under our assumptions all $G$-orbits on $\mathrm{Lie}(G)$ are smooth

Now suppose $\mathcal{F}$ to be separably closed and write $\mathcal{F}_{\mathrm{alg}}$ for an algebraic closure. Since $\mathrm{Lie}(R_Q)$ is an irreducible variety, the $\mathcal{F}$-points $\mathrm{Lie}(R_Q)(\mathcal{F})$ are *dense* in $\mathrm{Lie}(R_Q)$; see [Spr98, Lemma 11.2.5]. Thus, the unique dense $Q_{\mathcal{F}_{\mathrm{alg}}}$-orbit $\mathcal{O}$ on $\mathrm{Lie}(R_Q)_{\mathcal{F}_{\mathrm{alg}}}$ contains an element $X$ of $\mathrm{Lie}(R_Q)(\mathcal{F})$; since the orbit of $X$ is separable, it follows that $\mathcal{O}$ is defined over $\mathcal{F}$.

Now let $\mathcal{F}$ be arbitrary with separable closure $\mathcal{F}_{\mathrm{sep}}$. Since there is a *unique* open $Q_{\mathcal{F}_{\mathrm{sep}}}$-orbit $\mathcal{O}$ on $(\mathrm{Lie}\, R_u Q)_{\mathcal{F}_{\mathrm{sep}}}$, Galois descent shows that $\mathcal{O}$ is defined over $\mathcal{F}$.

Finally, there is always an $\mathcal{F}$-rational Richardson element. Indeed, when $\mathcal{F}$ is infinite an $\mathcal{F}$-open subset $\mathscr{U}$ of affine space $\mathrm{Lie}\, R_u P \simeq \mathbf{A}^N$ has an $\mathcal{F}$-rational point as soon as $\mathscr{U}(\mathcal{F}_{\mathrm{alg}})$ is non-empty, where $\mathcal{F}_{\mathrm{alg}}$ is an algebraic closure of $\mathcal{F}$. If instead $\mathcal{F}$ is finite, the existence of an $\mathcal{F}$-rational point follows from the Lang-Steinberg Theorem [Ser02, III §2]. □

*Remark* 3.3.5. The dense $Q$-orbit $\mathcal{O}$ on $\mathrm{Lie}(R_u Q)$ is called the *Richardson orbit* for $Q$, and an element $X \in \mathcal{O}(\mathcal{F}) \subset \mathrm{Lie}(R_u Q)(\mathcal{F})$ is called a *Richardson element* for $Q$.

If $G$ is semisimple, a parabolic subgroup $P \subset G$ is *distinguished* provided that $\dim P/R_u P = \dim R_u P - \dim \mathrm{der}(R_u P)$. when $G$ is reductive, a parabolic subgroup $P$ is distinguished provided that the intersection of $P$ with the derived group of $G$ is a distinguished parabolic. See [Car93, §5.8] and [Jan04, §4.10].

On the other hand, recall from Definition 1.1.1 that a nilpotent element $X \in \mathrm{Lie}(G)$ is *geometrically distinguished* provided that a maximal torus of $C_G(X)$ is central in $G$.

**Theorem 3.3.6.** *(a) Let $X \in \mathrm{Lie}(G)(\mathcal{F})$ be a geometrically distinguished nilpotent element and let $\phi : \mathbf{G}_m \to G$ be a cocharacter associated to $X$. Then $P = P_G(\phi)$ is a distinguished parabolic subgroup of $G$ and $X \in \mathrm{Lie}\, R_u P$ is a Richardson element.*
*(b) Let $Q$ be a distinguished parabolic subgroup of $G$. Then any Richardson element in $\mathrm{Lie}(R_u Q)$ is a geometrically distinguished nilpotent element.*

*Proof.* In each case it suffices to give the proof after extending scalars to an algebraic closure of $\mathcal{F}$. Now (a) follows from [Jan04]; see also [Pre03, Prop 2.5]. And (b) follows from [Car93, Cor. 5.2.4]. □

*Remark* 3.3.7. The preceding Theorem is a crucial part of the Bala-Carter Theorem parametrizing the geometric nilpotent orbits of G; it was first proved in good characteristic by K. Pommerening [Pom77; Pom80]. A more conceptual proof of the Bala-Carter Theorem was given later by Premet [Pre03].

3.4. **Balanced nilpotent sections for a reductive group scheme.** In this section, we are going to give the proofs of assertions (a) and (b) of Theorem 1.6.1.

For a field $\mathcal{F}$, by an $\mathcal{F}$-variety we mean an integral $\mathcal{F}$-scheme of finite type. If $f : V \to W$ is a morphism of varieties over $\mathcal{F}$, recall that $f$ is *dominant* if the image $f(V)$ is dense in $W$.

**Proposition 3.4.1.** *Let $R, S$ be finitely generated commutative $\mathcal{A}$-algebras which are* flat *- or equivalently (since $\mathcal{A}$ is a discrete valuation ring),* torsion free *- as $\mathcal{A}$-modules, and let $f : R \to S$ be an $\mathcal{A}$-algebra homomorphism. Assume that $f_k : R_k = R \otimes_{\mathcal{A}} k \to S_k = S \otimes_{\mathcal{A}} k$ is* injective*. Then $f$ is injective, and in particular $f_{\mathcal{K}} : R_{\mathcal{K}} \to S_{\mathcal{K}}$ is injective.*

*Proof.* Since $S$ is torsion free as an $\mathcal{A}$-module, also the image $B = \mathrm{im}(f)$ of $f$ is a torsion free and hence flat $\mathcal{A}$-module.

Writing $I = \ker f$, we have a short exact sequence of $\mathcal{A}$-modules

$$0 \to I \to R \xrightarrow{f} B \to 0.$$

Tensoring this sequence over $\mathcal{A}$ with $k$, we find an exact sequence

$$\mathrm{Tor}_1^{\mathcal{A}}(B, k) \to I \otimes_{\mathcal{A}} k \to R \otimes_{\mathcal{A}} k \xrightarrow{f_k} B \otimes_{\mathcal{A}} k \to 0.$$

Since $B$ is a flat $\mathcal{A}$-module, $\mathrm{Tor}_1^{\mathcal{A}}(B, k) = 0$. Since $f_k$ is injective, we deduce that $I/\pi I = I \otimes_{\mathcal{A}} k = 0$, where $\pi\mathcal{A}$ is the maximal ideal of DVR $\mathcal{A}$; put another way, we know that $I = \pi I$.

Now, $R$ is a finitely generated $\mathcal{A}$-algebra, hence $R$ is Noetherian. In particular, $I$ is a finitely generated $R$-module. In order to show that $I = 0$, it is enough to show that the localization $I_\mathfrak{m}$ is 0 for each maximal ideal $\mathfrak{m}$ of $R$.

Since $\pi \in \mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$, the condition $I = \pi I$ implies that $I_\mathfrak{m} = \mathfrak{m} I_\mathfrak{m}$; since $I_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module, Nakayama's Lemma implies that $I_\mathfrak{m} = 0$.

Conclude now that $I = 0$. This proves that $f$ - and *a fortiori* $f_\mathcal{K}$ - is injective, as required. □

We now return to the discrete valuation ring $\mathcal{A}$. Recall that an integral affine $\mathcal{A}$-scheme $Y = \operatorname{Spec}(\mathcal{A}[Y])$ is smooth over $\mathcal{A}$ if $\mathcal{A}[Y]$ is a flat $\mathcal{A}$-module which is finitely generated as $\mathcal{A}$-algebra, and if its fibers $Y_k$ and $Y_K$ "are" smooth varieties; see e.g. [BT84, (1.2.9)].

**Proposition 3.4.2.** *Let $X, Y$ be schemes which are affine, integral, smooth and of finite type over $\mathcal{A}$, and let $f : X \to Y$ be an $\mathcal{A}$-morphism. Suppose that the morphism $f_k : X_k \to Y_k$ obtained by base change is* dominant. *Then the morphism $f_\mathcal{K} : X_\mathcal{K} \to Y_\mathcal{K}$ is dominant, as well.*

*Proof.* Write $\mathcal{A}[X]$ and $\mathcal{A}[Y]$ for the affine coordinate rings of the schemes $X$ and $Y$, and write $k[X]$ and $k[Y]$ for the coordinate rings of the affine $k$-varieties $X_k$ and $Y_k$ obtained by base change; thus e.g. $k[X] = \mathcal{A}[X] \otimes_\mathcal{A} k$. Similarly, write $\mathcal{K}[X]$ and $\mathcal{K}[Y]$ for the coordinate rings of the affine $\mathcal{K}$-varieties $X_\mathcal{K}$ and $Y_\mathcal{K}$.

Since the morphism $f_k$ is dominant, [Stacks, Tag 0CC1] shows that the comorphism $f_k^* : k[Y] \to k[X]$ is *injective*. Since $X$ and $Y$ are smooth over $\mathcal{A}$, $\mathcal{A}[X]$ and $\mathcal{A}[Y]$ are free $\mathcal{A}$-modules. Proposition 3.4.1 permits us to conclude that the comorphisms $f^* : \mathcal{A}[Y] \to \mathcal{A}[X]$ and $f_\mathcal{K}^* : \mathcal{K}[Y] \to \mathcal{K}[X]$ are each injective, and now a second application of [Stacks, Tag 0CC1] shows that $f_\mathcal{K} : X_\mathcal{K} \to Y_\mathcal{K}$ is dominant. □

Now suppose that $\mathcal{G}$ is a reductive group scheme over $\mathcal{A}$ with connected fibers, and write $G = \mathcal{G}_\mathcal{K}$. Let us fix a parabolic subgroup scheme $\mathcal{Q}$ of $\mathcal{G}$. Recall from section 2.4 that there is a smooth subgroup scheme $\mathcal{R} = R_u\mathcal{Q}$ whose fibers are the unipotent radicals $R_u\mathcal{Q}_k$ and $R_u\mathcal{Q}_\mathcal{K}$.

**Proposition 3.4.3.** *Suppose that $X_0 \in \mathcal{R}_k$ is a Richardson element for $\mathcal{Q}_k$. Let $\mathcal{X} \in \mathcal{R}$ be any element with $\mathcal{X}_k = X_0$ – i.e. for which $\mathcal{X} \equiv X_0 \pmod{\pi\mathcal{R}}$. Then $\mathcal{X}_\mathcal{K} \in \mathcal{R}_K$ is a Richardson element for $\mathcal{Q}_K$.*

*Proof.* By abuse of notation, we also write $\mathcal{R}$ for the $\mathcal{A}$-scheme isomorphic to $\mathbf{A}^N$ whose $\mathcal{A}$-points identify with $\mathcal{R}$, where $N$ is the rank of $\mathcal{R}$ as $\mathcal{A}$-module. Consider the $\mathcal{A}$-morphism

$$\alpha : \mathcal{Q} \to \mathcal{R}$$

given by $\alpha(g) = \operatorname{Ad}(g)\mathcal{X}$.

The morphism $\alpha_k : \mathcal{Q}_k \to \mathcal{R}_k$ obtained by base change from $\alpha$ is then given by the rule $\alpha_k(g) = \operatorname{Ad}(g)X_0$; since $X_0$ is a representative of the dense $\mathcal{Q}_k$ orbit on $\mathcal{R}_k$, it follows that $\alpha_k$ is a dominant morphism.

Since $\mathcal{Q}$ and $\mathcal{R}$ are schemes which are affine, smooth, and of finite type over $\mathcal{A}$, it now follows from Proposition 3.4.2 that $\alpha_\mathcal{K} : \mathcal{Q}_\mathcal{K} \to \mathcal{R}_\mathcal{K}$ is dominant. In view of Proposition 3.3.4, this proves that $\mathcal{X}_\mathcal{K}$ is a Richardson element, as required. □

**Proposition 3.4.4.** *If $\mathcal{Q} \subset \mathcal{G}$ is a parabolic subgroup scheme, then $\mathcal{Q}_k$ is a distinguished parabolic subgroup of $\mathcal{G}_k$ if and only if $\mathcal{Q}_\mathcal{K}$ is a distinguished parabolic subgroup of $\mathcal{G}_\mathcal{K}$.*

*Proof.* Since $\mathcal{G}_k$ and $\mathcal{G}_\mathcal{K}$ are *geometrically standard* by assumption, this follows from the characterization of distinguished parabolics given in [Jan04, §4.10]. □

In order to prove the lifting results found in Section 1.6 of the introduction, we introduce a somewhat more precise notion. Consider a triple $(\mathcal{X}, \mathscr{S}, \phi)$ for which $\mathcal{X} \in \operatorname{Lie}(\mathcal{G})$, $\mathscr{S} \subset \mathcal{G}$ is a closed $\mathcal{A}$-subgroup scheme which is a maximal $\mathcal{A}$-torus (see Section 2.3), and $\phi : \mathbf{G}_m \to \mathcal{L}$ is an $\mathcal{A}$-homomorphism, where $\mathcal{L} = C_\mathcal{G}(\mathscr{S})$. We say that $(\mathcal{X}, \mathscr{S}, \phi)$ is a *balanced triple* if the following conditions hold:

(B1) $\mathcal{X}$ is balanced for the adjoint action of $\mathcal{G}$,
(B2) the cocharacter $\phi_\mathcal{K}$ is associated with the nilpotent element $\mathcal{X}_\mathcal{K} \in \operatorname{Lie}(\mathcal{G}_\mathcal{K})$ determined by $\mathcal{X}$,
(B3) the cocharacter $\phi_k$ is associated with the nilpotent element $\mathcal{X}_k \in \operatorname{Lie}(\mathcal{G}_k)$ determined by $\mathcal{X}$, and
(B4) $\mathcal{X}_k$ is geometrically distinguished in $\operatorname{Lie}(\mathcal{L}_k)$ and $\mathcal{X}_\mathcal{K}$ is geometrically distinguished in $\operatorname{Lie}(\mathcal{L}_\mathcal{K})$.

**Theorem 3.4.5.** *Fix a nilpotent element $X_0 \in \operatorname{Lie}(\mathcal{G}_k)$ and a maximal torus $S_0$ of the centralizer $C_{\mathcal{G}_k}(X_0)$. There is a maximal $\mathcal{A}$-torus $\mathscr{T}$ of $\mathcal{G}$ and a* balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ *such that $\mathscr{S} \subset \mathscr{T}$, $\mathscr{S}_k = S_0$ and $\mathcal{X}_k = X_0$.*

*Proof.* Write $L_0 = C_{\mathcal{G}_k}(S_0)$. Using Proposition 3.3.2, we choose a cocharacter $\phi_0$ of $L_0$ associated with $X_0$; thus also $\phi_0$ is associated with $X_0$ in $\mathrm{Lie}(\mathcal{G}_k)$; cf. Theorem A.1.

Using Corollary 2.3.4 and Theorem 2.3.1 we may find $\mathcal{A}$-tori $\mathscr{S} \subset \mathscr{T} \subset \mathcal{G}$ together with an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathscr{T}$ for which $\mathscr{T}$ is a maximal torus of $\mathcal{G}$, $\mathscr{S}_k = S_0$, and $\phi_k : \mathbf{G}_m \to \mathscr{T}_k \subset \mathcal{G}_k$ coincides with $\phi_0$.

Set $\mathcal{L} = C_{\mathcal{G}}(\mathscr{S})$; according to Proposition 2.4.1, $\mathcal{L}$ is a reductive group scheme with connected fibers. Moreover, since $\mathcal{G}$ is a standard reductive group scheme, also $\mathcal{L}$ is a standard reductive group scheme.

By definition $\phi_0$ takes values in the derived group $\mathrm{der}(\mathcal{L}_k) = \mathrm{der}(L_0)$. Using Proposition 2.3.5 we may find an $\mathcal{A}$-morphism $\phi : \mathbf{G}_m \to \mathcal{L}$ such that $\phi_0 = \phi_k$ and such that $\phi_{\mathcal{K}}$ takes values in the derived group $\mathrm{der}(\mathcal{L}_{\mathcal{K}})$. Note that (B3) holds by construction.

Now let $Q = P_{\mathcal{L}}(\phi)$ and $P = P_{\mathcal{G}}(\phi)$ be the parabolic subgroup schemes of $\mathcal{L}$ and $\mathcal{G}$ determined by $\phi$ as in Proposition 2.4.3.

Since a maximal torus of $C_{\mathcal{L}_k}(X_0)$ is central in $\mathcal{L}$, $X_0$ is distinguished in $\mathrm{Lie}(\mathcal{L}_k)$. It follows from Theorem 3.3.6 that $X_0 \in \mathrm{Lie}(R_u Q_k)$ is a Richardson element and that $Q_k$ is a distinguished parabolic subgroup of $\mathcal{L}_k$. It follows from Proposition 3.4.4 that $Q_{\mathcal{K}}$ is a distinguished parabolic subgroup of $\mathcal{L}_{\mathcal{K}}$. Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{L})(\phi;2) \subset \mathrm{Lie}(R_u Q)$ by any element with $\mathcal{X}_k = X_0$. It follows from Proposition 3.4.3 that $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(R_u Q_{\mathcal{K}})$ is Richardson. Since $Q_{\mathcal{K}}$ is a distinguished parabolic, a second application of Theorem 3.3.6 shows that $\mathcal{X}_{\mathcal{K}}$ is distinguished in $\mathrm{Lie}(\mathcal{L}_{\mathcal{K}})$; this confirms condition (B4). Since $\mathcal{X}_{\mathcal{K}} \in \mathrm{Lie}(\mathcal{L}_{\mathcal{K}})(\phi_{\mathcal{K}};2)$ and since the image of $\phi_{\mathcal{K}}$ lies in $\mathrm{der}(\mathcal{L}_{\mathcal{K}})$, it follows that $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$ in $\mathcal{L}_{\mathcal{K}}$. It now follows from the Theorem proved in the appendix to this paper – see Theorem A.1 – that $\phi_{\mathcal{K}}$ is a cocharacter associated to $\mathcal{X}_{\mathcal{K}}$ in $\mathcal{G}_{\mathcal{K}}$, as well. Thus condition (B2) holds

Finally, since $\mathcal{G}_k$ and $\mathcal{G}_{\mathcal{K}}$ are geometrically standard reductive groups, the centralizers $C_{\mathcal{G}_k}(\mathcal{X}_k)$ and $C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ are smooth – see Proposition 2.1.4. Since $\phi_{\mathcal{K}}$ is associated with $\mathcal{X}_{\mathcal{K}}$ and $\phi_k$ is associated with $\mathcal{X}_k$, it follows from Proposition 3.3.3 that the dimension of $C_{\mathcal{G}_k}(\mathcal{X}_k)$ coincides with that of $C_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$. This shows that $\mathcal{X}$ is indeed balanced; this confirms condition (B1) and completes the proof of the Theorem. □

We observe that the preceding result indeed confirms assertions (a) and (b) of the Theorem formulated in Section 1.6 of the introduction to this paper.

*Proof of Theorem 1.6.1(a) and (b).* Let $X \in \mathrm{Lie}(\mathcal{G}_k)$ be nilpotent, and choose a maximal $k$-torus $S$ of $C_{\mathcal{G}_k}(X)$. The preceding Theorem then yields a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$. Now (a) of Theorem 1.6.1 is confirmed. Now if $X$ is geometrically distinguished for $\mathcal{G}_k$, then condition (B4) shows that $\mathcal{X}_{\mathcal{K}}$ is geometrically distinguished for $\mathcal{G}_{\mathcal{K}}$; this confirms (b). □

3.5. **The centralizer of a balanced section.** We will later require a property of the centralizer of a balanced section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$, which we now formulate and prove.

Let $G$ be an unramified reductive group over $\mathcal{K}$ with reductive $\mathcal{A}$-model $\mathcal{G}$, suppose that $\mathcal{G}$ is a standard reductive group scheme, and let $(\mathcal{X}, \mathscr{S}, \psi)$ be a balanced triple as in Section 3.4. We require some results about the centralizer group scheme $C = C_{\mathcal{M}}(\mathcal{X})$.

Recall from Corollary 3.1.5 that since $\mathcal{X}$ is balanced, $C$ contains an open subgroup scheme $\mathcal{H}$ with the following properties:

(B1) $\mathcal{H}$ is smooth and affine over $\mathcal{A}$,
(B2) the fibers $\mathcal{H}_{\mathcal{K}}$ and $\mathcal{H}_k$ coincide with the identity component of $C_{\mathcal{K}}$ respectively of $C_k$.

**Proposition 3.5.1.** *$\mathcal{H}$ contains a subgroup scheme $\mathcal{M}$ with the following properties:*

*(a) $\mathcal{H}$ is reductive over $\mathcal{A}$,*
*(b) $\mathcal{M}_k$ is a Levi factor of $\mathcal{H}_k = C_k^0$, and*
*(c) $\mathcal{M}_{\mathcal{K}}$ is a Levi factor of $\mathcal{H}_{\mathcal{K}} = C_{\mathcal{K}}^0$.*

*Proof.* Note that $\mathcal{H}$ is normalized by the action of the image of $\phi$; see Remark 3.1.6. Set $\mathcal{M} = C_{\mathcal{H}}(\phi)$ for the centralizer subgroup scheme of the image of $\phi$ in $\mathcal{H}$; more precisely, the group scheme $C_{\mathcal{H}}(\phi)$ is the fixed-point subscheme of $\mathcal{H}$ for the action of the image of $\phi$. According to [$\mathrm{SGA3}_{\mathrm{II}}$, Exp. XII, Prop. 9.2 and Cor 9.8], $\mathcal{M}$ is a closed subscheme of $\mathcal{H}$ which is smooth over $\mathcal{A}$. Moreover, $\mathcal{M}$ is affine over $\mathcal{A}$, e.g. by Proposition 3.1.1.

Now, $\mathcal{M}_{\mathcal{K}}$ is the centralizer in $C^o_{\mathcal{G}_{\mathcal{K}}}(\mathcal{X}_{\mathcal{K}})$ of the image of $\phi_{\mathcal{K}}$ and $\mathcal{M}_k$ is the centralizer in $C^o_{\mathcal{G}_k}(\mathcal{X}_k)$ of the image of $\phi_k$. Thus it follows from Proposition 3.3.2 that for $\mathcal{F} \in \{\mathcal{K}, k\}$, $\mathcal{M}_{\mathcal{F}}$ is a Levi factor of $C^0_{\mathcal{G}_{\mathcal{F}}}(\mathcal{X}_{\mathcal{F}})$.

In particular, $\mathcal{M}$ is a smooth and affine $\mathcal{A}$-group scheme with connected and reductive fibers, hence $\mathcal{M}$ is reductive [SGA3$_{III}$, Exp. XIX Defn 2.7]. □

**Proposition 3.5.2.** *There is an $\mathcal{A}$-torus $\mathscr{T}$ of $\mathcal{G}$ centralizing $\mathcal{X}$ such that $\mathscr{T}_{\mathcal{F}}$ is a maximal torus of $C_{\mathcal{F}}$ for $\mathcal{F} = k, \mathcal{K}$. In particular, the centralizer $C_G(\mathcal{X}_{\mathcal{K}})$ has a maximal torus which splits over an unramified extension of $\mathcal{K}$.*

*Proof.* Let $\mathcal{M} \subset \mathcal{H}$ be the open subgroup scheme of the centralizer as in Proposition 3.5.1.

Applying Corollary 2.3.4 we see that the reductive $\mathcal{A}$-group scheme $\mathcal{M} \subset \mathcal{H}$ has a maximal torus $\mathscr{T}$. According to Corollary 2.3.2 that torus splits over an unramified extension $\mathcal{B} \supset \mathcal{A}$, and in particular the maximal torus $\mathscr{T}_{\mathcal{K}}$ of $\mathcal{M}_{\mathcal{K}} = C^0_{\mathcal{K}}$ splits over the field of fractions of $\mathcal{B}$, an unramified extension of $\mathcal{K}$. □

## 4. $SL_2$-HOMOMORPHISMS

As discussed in in Section 1.7, some of the proofs of the main results of this paper depend on the existence of $SL_2$-homomorphisms determined by nilpotent elements. In this section, we confirm these required results.

Let us fix some notations. For a commutative ring $A$, consider the semisimple $A$-group scheme $SL_{2,A}$. Write $E_A$ and $F_A$ for the nilpotent sections

$$E_A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Lie}(SL_{2/A}),$$

and write $\mathscr{D} = \mathscr{D}_A \subset SL_{2,A}$ for the diagonal (maximal) torus; thus $\mathscr{D}_A \simeq \mathbf{G}_{m/A}$.

We identify the character groups

$$X^*(\mathscr{D}) = X^*(\mathscr{D}_k) = X^*(\mathscr{D}_{\mathcal{K}})$$

with $\mathbf{Z}$. Any $\mathscr{D}$-module $\mathcal{M}$ is completely reducible and may be written as a direct sum

$$(\heartsuit) \quad \mathcal{M} = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}_n$$

of its weight spaces; see [Jan03, p. I.2.11].

4.1. **Modules for $SL_2$ and lifting.** We begin by formulating some generalities about representations of split reductive groups over the complete DVR $\mathcal{A}$.

First, consider an $\mathcal{A}$-algebra $R$ which is free and of finite rank as an $\mathcal{A}$-module, and write $R_k = R \otimes_{\mathcal{A}} k$ and $R_{\mathcal{K}} = R \otimes_{\mathcal{A}} \mathcal{K}$.

**Proposition 4.1.1.** *Let $V$ be a finitely generated $R_k$-module, and suppose that $\mathrm{Ext}^i_{R_k}(V, V) = 0$ for $i = 1, 2$. Then there is an $R$-module $\mathscr{V}$, unique up to isomorphism of $R$-modules, such that $\mathscr{V}$ is free of rank $\dim_k V$ and $V \simeq \mathscr{V} \otimes_{\mathcal{A}} k$ as $R_k$-module.*

*Proof.* This follows from [McN00, Prop. 3.1.2]. □

Consider a split reductive group scheme $\mathcal{G}$ over the complete DVR $\mathcal{A}$. A representation of $\mathcal{G}$ ("$\mathcal{G}$-module") is of course a *co-module* for the Hopf algebra $\mathcal{A}[\mathcal{G}]$ (the coordinate algebra of $\mathcal{G}$).

Fix a split maximal torus $\mathscr{T}$ of $\mathcal{G}$ and write $X^* = X^*(\mathscr{T})$ for the group of its characters. Any $\mathcal{G}$-module $V$ is a direct sum $V = \bigoplus_{\lambda \in X^*} V_\lambda$ of its $\mathscr{T}$-weight spaces.

For a *saturated set of weights* $\pi \subset X^*$ – see [Bou02, VI Exerc. I.23] – and for $\Lambda \in \{\mathcal{A}, k, \mathcal{K}\}$, consider the full subcategory $\mathcal{C}_{\pi,\Lambda}$ of the category of $\mathcal{G}_\Lambda$-modules whose objects are those $\mathcal{G}_\Lambda$-modules $V$ for which $V_\lambda \neq 0 \implies \lambda \in \pi$.

Donkin [Don86] has introduced an $\mathcal{A}$-algebra $S_\pi$ – a *generalized Schur algebra* – which is finitely generated and free as an $\mathcal{A}$-module and for which the category of $S_{\pi,\Lambda}$-modules is naturally equivalent to $\mathcal{C}_{\pi,\Lambda}$ where $S_{\pi,\Lambda} = S_\pi \otimes_{\mathcal{A}} \Lambda$. See also [McN00, §5.2].

As a consequence, we find the following:

**Proposition 4.1.2.** *Let $V$ be an finite dimensional algebraic $\mathcal{G}_k$-representation for which $\mathrm{Ext}^i_{\mathcal{G}_k}(V, V) = 0$ for $i = 1, 2$. Then there is a representation $\mathscr{V}$ for the group scheme $\mathcal{G}$ such that $\mathscr{V}$ is a free $\mathcal{A}$-module, and $V \simeq \mathscr{V} \otimes_{\mathcal{A}} k$ as $\mathcal{G}_k$-modules. Moreover, $\mathscr{V}$ is uniquely determined by these properties, up to isomorphism of $\mathcal{G}$-modules.*

*Proof.* Indeed, choose a saturated set of weights $\pi$ large enough to contain all weights $\lambda$ for which $V_\lambda \neq 0$. Using Donkin's result, we may view V as a module for $S_{\pi,k}$. Our assumptions now permit application of Proposition 4.1.1. □

**Proposition 4.1.3.** *Let $\mathcal{M}$ and $\mathscr{N}$ be $\mathcal{G}$-modules that are free and of finite rank as $\mathcal{A}$-modules. Suppose that*

$$\operatorname{Ext}^1_{\mathcal{G}_k}(\mathcal{M}_k, \mathscr{N}_k) = 0.$$

*If $\phi_0 : \mathcal{M}_k \to \mathscr{N}_k$ is an isomorphism of $\mathcal{G}_k$-modules, there is an isomorphism $\Phi : \mathcal{M} \to \mathscr{N}$ of $\mathcal{G}$-modules for which $\phi_0 = \Phi_k$.*

*Proof.* Under our assumption on $\operatorname{Ext}^1$, it follows from [McN00, Prop. 3.3.1, Prop 3.3.2] that the natural mapping $(\Phi \mapsto \Phi_k) : \operatorname{Hom}_{\mathcal{G}}(\mathcal{M}, \mathcal{L}) \to \operatorname{Hom}_{\mathcal{G}_k}(\mathcal{M}_k, \mathcal{L}_k)$ is surjective. Take $\Phi : \mathcal{M} \to \mathcal{L}$ such that $\Phi_k = \phi_0$; it follows from the Nakayama lemma that $\Phi$ is an isomorphism, as required. □

We are mainly going to apply the preceding results to $\mathrm{SL}_{2,\mathcal{A}}$, so we now specialize our discussion a bit. In this case, $\mathscr{D}$ is a split maximal torus and its character group may be identified with the free abelian group $\mathbf{Z}$.

Let $\mathcal{F}$ be a field of characteristic $p \geqslant 0$. For a representation V of $\mathrm{SL}_{2,\mathcal{F}}$, write $V = \bigoplus_n V_n$ as a direct sum of $\mathscr{D}$-weight spaces. We say that V is *restricted* if either $p = 0$ or if $V_n \neq 0 \implies |n| < p$.

**Proposition 4.1.4.** *(a) A restricted $\mathrm{SL}_{2,\mathcal{F}}$-representation is semisimple.*
*(b) Any restricted representation V is the internal direct sum (as $\mathscr{D}$-module) of $\ker(F : V \to V)$ and $E.V$.*
*(c) If $V, W$ are restricted $\mathrm{SL}_{2,\mathcal{F}}$-representations, then*

$$\operatorname{Ext}^i_{\mathrm{SL}_{2,\mathcal{F}}}(V, W) = 0, \quad i \geqslant 1.$$

*Proof.* Recall [Jan03, §II.2] that the simple modules $L_{\mathcal{F}}(n)$ for $\mathrm{SL}_{2,\mathcal{F}}$ are indexed by their highest weight $n \in \mathbf{Z}_{\geqslant 0}$, where we have identified $X^*(\mathscr{D}_{\mathcal{F}}) = \mathbf{Z}$. A simple module is then restricted if and only if $n < p$. It follows from the *linkage principle* [Jan03, Cor. II.6.7] that $\operatorname{Ext}^i_{\mathrm{SL}_{2,\mathcal{F}}}(L, L') = 0$ for $i \geqslant 1$ and for any two restricted simple modules. (a) and (c) are immediate consequences of this statement. Since (b) holds for each restricted simple module $L(n)$, it holds for V by assertion (a). □

Let $\mathscr{V}$ be an $\mathrm{SL}_{2,\mathcal{A}}$-module. We will say that $\mathscr{V}$ is restricted if $\mathscr{V}$ is free of finite rank as $\mathcal{A}$-module and if $\mathscr{V}_k$ is a restricted module for $\mathrm{SL}_{2,k}$. Of course, $\mathscr{V}$ is restricted if and only if $n \geqslant p \implies \mathscr{V}_n = 0$ where $\mathscr{V}_n$ are the weight spaces in the decomposition (♡) above, where p is the residue characteristic.

Combining Proposition 4.1.2 and Proposition 4.1.4, we obtain:

**Proposition 4.1.5.** *Let V be a restricted representation for $\mathrm{SL}_{2,k}$. Then there is a representation $\mathscr{V}$ for $\mathrm{SL}_{2,\mathcal{A}}$ for which $\mathscr{V}$ is a free $\mathcal{A}$-module and $V \simeq \mathscr{V} \otimes_{\mathcal{A}} k$ as $\mathrm{SL}_{2,k}$-modules; thus $\mathscr{V}$ is restricted. Moreover, $\mathscr{V}$ is uniquely determined up to isomorphism of $\mathrm{SL}_{2,\mathcal{A}}$-modules.*

For $0 \leqslant n < p$, consider the simple $\mathrm{SL}_{2,k}$-module $L(n)$. Since $L(n)$ is restricted, Proposition 4.1.5 yields a restricted $\mathrm{SL}_{2,\mathcal{A}}$-module $\mathcal{L}(n)$ with $\mathcal{L}(n) \otimes k \simeq L(n)$. Since $\mathcal{L}(n) \otimes \mathcal{K}$ is restricted - hence semisimple - and has 1-dimensional weight spaces, it is immediate that $\mathcal{L}(n) \otimes \mathcal{K}$ is a simple $\mathrm{SL}_{2,\mathcal{A}}$-module, as well.

We now find the following:

**Proposition 4.1.6.** *If $\mathscr{V}$ is a restricted $\mathrm{SL}_{2,\mathcal{A}}$-module, then $\mathscr{V} \simeq \bigoplus_i \mathcal{L}(n_i)$ for integers $0 \leqslant n_i < p$ which are uniquely determined up to order.*

*Proof.* Since $\mathscr{V}_k$ is a restricted semisimple $\mathrm{SL}_{2,k}$-module, there is an isomorphism $\mathscr{V}_k \simeq \bigoplus_i L(n_i)$ of $\mathrm{SL}_{2,k}$-modules, where $0 \leqslant n_i < p$ for each i. It now follows from Proposition 4.1.3 that there is an isomorphism of $\mathrm{SL}_{2,\mathcal{A}}$-modules $\mathscr{V} \simeq \bigoplus_i \mathcal{L}(n_i)$. □

**Proposition 4.1.7.** *If $\mathscr{V}$ is a restricted module, we have:*
*(a) $\dim_k \ker(E_k : \mathscr{V}_k \to \mathscr{V}_k) = \dim_{\mathcal{K}} \ker(E_{\mathcal{K}} : \mathscr{V}_{\mathcal{K}} \to \mathscr{V}_{\mathcal{K}})$.*
*(b) $\mathscr{V}$ is the internal direct sum $\ker(F) \oplus E.\mathscr{V}$ as $\mathscr{D}_{\mathcal{A}}$-modules.*

*Proof.* In view of Proposition 4.1.6, it suffices to confirm (a) and (b) when $\mathscr{V} = \mathcal{L}(n)$ for each $0 \leqslant n < p$.

Let $\mathcal{F}$ be either k or $\mathcal{K}$. Since $\mathcal{L}(n) \otimes \mathcal{F}$ is the restricted, simple $\mathrm{SL}_{2,\mathcal{F}}$-module $L_{\mathcal{F}}(n)$ of highest weight n, it is clear that $\ker(E_{\mathcal{F}} : \mathcal{L}(n) \otimes \mathcal{F} \to \mathcal{L}(n) \otimes \mathcal{F})$ is 1 dimensional and is spanned by a $\mathscr{D}_{\mathcal{F}}$-weight vector of weight n. Now (a) is confirmed.

The same argument also show that $\ker(F_{\mathcal{F}} : \mathcal{L}(n) \otimes \mathcal{F} \to \mathcal{L}(n) \otimes \mathcal{F})$ is 1-dimensional and is spanned by a $\mathscr{D}_{\mathcal{F}}$-weight vector of weight $-n$.

Finally, since $\ker(E_{\mathcal{F}})$ is 1 dimensional, it is clear that $E.\mathcal{L}(n) \otimes \mathcal{F}$ has codimension 1, and coincides with $\bigoplus_{i=-n}^{n-2} \mathcal{L}(i) \otimes \mathcal{F}$. Now (b) follows from the Nakayama Lemma. □

For any $n \in \mathbf{Z}_{\geqslant 0}$, write

$$L(n) = L_{\mathcal{F}}(n), \quad H^0(n) = H^0_{\mathcal{F}}(n), \quad V(n) = V_{\mathcal{F}}(n) = H^0(n)^{\vee}, \quad T(n) = T_{\mathcal{F}}(n)$$

respectively for the simple, standard, Weyl and tilting $SL_{2,\mathcal{F}}$-module of highest weight $n$; see [Jan03, §II.2]. If $n < p$ so that $L(n)$ is restricted, then $T(n) = V(n) = H^0(n) = L(n)$, but in general these modules do not coincide.

An $SL_{2,\mathcal{F}}$-module $T$ is said to be a tilting module provided that $T$ and the dual module $T^{\vee}$ both have a filtration by standard modules $H^0(n)$.

We have the following characterization of tilting modules:

**Proposition 4.1.8.** *Let $T$ be an $SL_{2,\mathcal{F}}$-module. The following are equivalent:*
*(a) $T$ is a tilting module,*
*(b) $\mathrm{Ext}^1_{SL_{2,\mathcal{F}}}(V(n), T) = \mathrm{Ext}^1_{SL_{2,\mathcal{F}}}(T, H^0(n)) = 0$ for all $n \geqslant 0$,*
*(c) $\mathrm{Ext}^i_{SL_{2,\mathcal{F}}}(V(n), T) = \mathrm{Ext}^i_{SL_{2,\mathcal{F}}}(T, H^0(n)) = 0$ for all $n \geqslant 0$ and all $i \geqslant 1$.*

*Proof.* This result is confirmed e.g. in [Jan03, Prop. E.1]. □

**Proposition 4.1.9.** *Let $T$ be a tilting module for $SL_{2,\mathcal{F}}$. Then $T \simeq \bigoplus_i T(n_i)$ for certain integers $n_i \geqslant 0$.*

*Proof.* Indeed, by the Krull-Schmidt theorem we may write $T$ as a direct sum of indecomposable $SL_{2,\mathcal{F}}$-modules. Now Proposition 4.1.8 shows that any direct summand of a tilting module is again tilting. Thus the Proposition follows from the description of indecomposable tilting modules; cf. [Jan03, Prop. E.6]. □

In particular, in the setting of the local field $\mathcal{K}$, if $T$ is a tilting module for $SL_{2,k}$, Proposition 4.1.2 guarantees that there is a module $\mathscr{T}$ for $SL_{2,\mathcal{A}}$ which is free of finite rank as $\mathcal{A}$-module for which $\mathscr{T} \otimes k \simeq T$.

4.2. **Optimal $SL_2$-homomorphisms over a field.** Consider a geometrically standard reductive group $G$ over a field $\mathcal{F}$ of characteristic $p \geqslant 0$.

For a nilpotent element $X \in \mathrm{Lie}(G)$ recall that according to Proposition 3.3.2 we may choose a cocharacter $\phi$ of $G$ associated with $X$. Write $\mathrm{Lie}(G)(i) = \mathrm{Lie}(G)(\phi; i)$ for the $i$-weight space in $\mathrm{Lie}(G)$ for the adjoint action of the image of $\phi$.

**Proposition 4.2.1.** *Let as usual $h$ denote the Coxeter number of $G$. Suppose that $X_1 \in \mathrm{Lie}(G)(\mathcal{F})$ is a* regular *nilpotent element, and let $\phi_1$ be an associated cocharacter. Then $\mathrm{Lie}(G)(i) \neq 0 \implies |i| \leqslant 2h - 2$. In particular, $\mathrm{ad}(Y)^{2h-2} = 0$ for every nilpotent $Y \in \mathrm{Lie}(G)$.*

*Proof.* The conclusion of the Proposition is unaffected by scalar extension, thus we may suppose that $G$ is split and that $X_1$ is the sum of all the simple root vectors. In that case, one can explicitly identify the cocharacter $\phi_1$ and one knows that $\mathrm{Lie}(G) = \displaystyle\bigoplus_{i=-h+1}^{h-1} \mathrm{Lie}(G)(2i)$. This confirms that $\mathrm{ad}(X)^{2h-2} = 0$. It follows that the morphism $Y \mapsto \mathrm{ad}(Y)^{2h-2}$ defined on the nilpotent cone vanishes on the open, dense set of regular nilpotent elements, and hence this morphism is identically zero. □

If $p > 0$ recall that $\mathrm{Lie}(G)$ is a $p$-Lie algebra, so there is a $p$-operation $Y \mapsto Y^{[p]}$ on $\mathrm{Lie}(G)$. An element $Y$ is nilpotent if and only if iterated application of the $p$-operation annihilates $Y$ – i.e. $Y^{[p]^j} = 0$ for some $j \geqslant 1$. For notational convenience, when the characteristic of $\mathcal{F}$ is zero, we *define* $Y^{[p]}$ to be 0, for *nilpotent* $Y \in \mathrm{Lie}(G)$.

**Theorem 4.2.2.** *Let $X \in \mathrm{Lie}(G)$ be nilpotent with $X^{[p]} = 0$, and choose a cocharacter $\phi$ associated with $X$. Then there is a unique $\mathcal{F}$-homomorphism $\Psi : SL_{2,\mathcal{F}} \to G$ such that $d\Psi(E_{\mathcal{F}}) = X$ and $\Psi_{|\mathscr{D}_{\mathcal{F}}} = \phi$.*

*Proof.* For a standard reductive group over $\mathcal{F}$, this is proved in [McN05, Theorem 47]; see Remark 2.1.2. For geometrically standard $G$ the result follows by Galois descent from the result for standard $G$. □

*Definition* 4.2.3. A homomorphism $\Psi : SL_2 \to G$ is said to be *optimal* if $\phi = \Psi_{|\mathscr{D}_{\mathcal{F}}}$ is a cocharacter associated to the nilpotent element $X = d\Psi(E_{\mathcal{F}})$.

Thus Theorem 4.2.2, provides an optimal homomorphism associated with any nilpotent element $X$.

Write $h$ for the Coxeter number of $G$. We record the following:

**Proposition 4.2.4.** *Let $X \in \mathrm{Lie}(G)$ be nilpotent and let $\phi$ be a cocharacter associated with $X$. Then*

*(a)* $X^{[p]} = 0$ *if and only if* $\mathrm{Lie}(G)(\phi; i) = 0$ *for all* $i \geqslant 2p$.

*(b) If* $p \geqslant h$, *then* $X^{[p]} = 0$.

*Proof.* When $G$ is standard, (a) follows from [McN05, Prop. 24] (again, see Remark 2.1.2). When $G$ is geometrically standard, (a) follows by Galois descent.

For (b), it suffices to prove the result after extending the ground field. Thus we may and will suppose that $G$ is split reductive over $\mathcal{F}$ and we may choose a regular nilpotent element $X_1 \in \mathrm{Lie}(G)(\mathcal{F})$. If $\phi_1$ is a cocharacter associated to $X_1$, then $X_1^{[p]} \in \mathrm{Lie}(G)(\phi_1; 2p)$. On the other hand, if $p \geqslant h$ Proposition 4.2.1 shows that $\mathrm{Lie}(G)(\phi_1; 2p) = 0$. We conclude that $X_1^{[p]} = 0$. Thus, the morphism $Y \mapsto Y^{[p]}$ from the nilpotent variety to itself vanishes on the dense, open subvariety of regular elements and so is identically zero. This proves (b). □

The following result is the basis for [McN05, Theorem 35]; see also [Sei00, §2].

**Proposition 4.2.5.** *Suppose that $p > 0$, let $X \in \mathrm{Lie}(G)$ be nilpotent with $X^{[p]} = 0$, choose a cocharacter $\phi$ associated to $X$, and let $\psi : SL_{2,\mathcal{F}} \to G$ be the optimal $SL_2$-homomorphism determined by $X$ and $\phi$ as in Theorem 4.2.2. Then as a representation for $SL_2$, the adjoint representation $\mathrm{Lie}(G)$ is a* tilting module. *Moreover, as $SL_2$-module,*

$$\mathrm{Lie}(G) \simeq \bigoplus_{0 \leqslant n \leqslant 2p-2} T(n)^{e(n)}$$

*for suitable multiplicities $e(n) \geqslant 0$.*

*Proof.* Since $SL_{2,\mathcal{F}}$ is split reductive over $\mathcal{F}$, two semisimple $SL_{2,\mathcal{F}}$-modules which become isomorphic after scalar extension of $\mathcal{F}$ are already isomorphic over $\mathcal{F}$. Thus it suffices to prove the result after scalar extension, so we may and will suppose suppose $G$ to be split over $\mathcal{F}$.

Since $G$ is split, it arises by base change from a split reductive group scheme over the discrete valuation ring $\mathbf{Z}_p$, the $p$-adic integers. Write $\Lambda$ for a complete discrete valuation ring of characteristic 0 with residue field $\mathcal{F}$ – see [Bou06, IX.2.3 Prop 5] for the existence of such a ring. Since the complete DVR $\Lambda$ is an extension of the $p$-adic integers, it follows that $G$ arises by base change from a split reductive group scheme $\mathcal{G}$ over $\Lambda$.

Now argue as in [McN05, Lemma 28] – or just use Theorem 3.4.5 – to find a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ with $\mathcal{X}_{\mathcal{F}} = X$ for which $\mathscr{S}_{\mathcal{F}}$ is a maximal torus of the centralizer $C_G(X)$. It now follows from [McN03b, Theorem 13] – or from Proposition 4.3.3 below – that there is a $\Lambda$-homomorphism $\Psi : SL_{2,\Lambda} \to \mathcal{G}$ for which $\Phi = \Psi_{\mathcal{F}}$. Moreover, Proposition 4.2.4(b) shows that (♣) $\quad |i| \geqslant 2p \implies \mathfrak{g}(\phi; i) = 0$.

Observe that (♣) implies that all weights $n$ of the $SL_{2,\Lambda}$-represent $\mathrm{Lie}(G)$ satisfy $n < 2p$. This condition permits us to apply [McN05, Prop. 34] (or [Sei00, Prop. 4.2]), and thus we conclude that $\mathrm{Lie}(G)$ is indeed a tilting module for $SL_{2,\mathcal{F}}$ via the homomorphism $\Phi$.

Thus as a module for $SL_{2,\mathcal{F}}$, $\mathrm{Lie}(G)$ is a direct sum of indecomposable tilting modules $T(d)$; now (♣) guarantees that each tilting summand $T(d)$has highest weight $d \leqslant 2p-2$, as required. □

**Proposition 4.2.6.** *Suppose that the characteristic $p$ of $\mathcal{F}$ satisfies $p = 0$ or $p > 2h-2$, and let $X \in \mathrm{Lie}(G)$ be nilpotent. Then $X^{[p]} = 0$ and if $\Psi : SL_2 \to G$ is an optimal homomorphism for which $d\Psi(E_{\mathcal{F}}) = X$, then the adjoint representation $\mathrm{Lie}(G)$ is restricted (and in particular semisimple) as $SL_{2,\mathcal{F}}$-module. Moreover, if the simple $SL_{2,\mathcal{F}}$-module $L(n)$ appears as a submodule of $\mathrm{Lie}(G)$, then $n \leqslant 2h-2$.*

*Proof.* Proposition 4.2.1 shows that $\mathrm{ad}(X)^{2h-2} = 0$. If $p > 2h-2$, then $\mathrm{ad}(X)^{p-1} = 0$; this confirms the condition $X^{[p]} = 0$.

Now use Theorem 4.2.2 to find an optimal homomorphism $\Psi : SL_2 \to G$ with $d\Psi(E_{\mathcal{F}}) = X$. If $p = 0$, every $SL_{2,\mathcal{F}}$-module is semisimple, so suppose for the moment that $p > 0$. According to Proposition 4.2.5, the adjoint representation $\mathrm{Lie}(G)$ viewed as an $SL_2$-module via $\Psi$ is a direct sum of tilting modules $T(d)$

for $d \leqslant 2p-2$. For any $p \leqslant d \leqslant 2p-2$, the $SL_2$-module $T(d)$ has $\dim T(d) = 2p$, and $E_{\mathcal{F}}$ acts on $T(d)$ with two Jordan blocks each of size $p$; see [McN03a, Prop. 5] or the description in [Sei00, Lemma 2.3]. Since $\mathrm{ad}(X)^{p-1} = 0$, $\mathrm{ad}(X)$ has no Jordan blocks of size $p$. It follows that the only tilting modules that can appear in the decomposition of $\mathrm{Lie}(G)$ as $SL_2$-module are those of the form $T(d)$ with $d < p-1$. But those tilting modules are restricted – i.e. $T(d) = L(d)$ for $0 \leqslant d < p$; this confirms that $\mathrm{Lie}(G)$ is a restricted module for $SL_{2,\mathcal{F}}$.

To conclude the proof, we return to the general case in which $p \geqslant 0$. Since $E_{\mathcal{F}}$ has a Jordan block of size $m$ on a restricted simple $SL_{2,\mathcal{F}}$-module $L(m)$, and since $\mathrm{ad}(X)^{2h-2} = 0$, the simple $SL_{2,\mathcal{F}}$-submodules $L(n)$ of the restricted semisimple module $\mathrm{Lie}(G)$ must satisfy $n \leqslant 2h-2$. □

4.3. **Optimal $SL_2$-homomorphisms over $\mathcal{A}$.** The main goal of this section is the proof of Theorem 4.3.6; this will establish Theorem 1.7.1 from the introduction. As we'll observe in more detail below, the proofs of the results found in this section amount more-or-less to a recapitulation of arguments given in [McN03b] and [McN05]; however, the formulation given here isn't found in those references.

Let $G$ be an unramified reductive group over $\mathcal{K}$ with reductive model $\mathcal{G}$ over $\mathcal{A}$, and suppose that $\mathcal{G}$ is a standard reductive group scheme. Fix a balanced triple for $\mathcal{G}$

$$(\heartsuit) \qquad (\mathcal{X}, \mathscr{S}, \phi)$$

as in Section 1.6. Recall that $\phi_k$ is a cocharacter associated with $\mathcal{X}_k$ and that $\phi_{\mathcal{K}}$ is a cocharacter associated with $\mathcal{X}_{\mathcal{K}}$. Write $p$ for the characteristic of the residue field $k$.

**Lemma 4.3.1.** *If $(\mathcal{X}_k)^{[p]} = 0$, then $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$.*

*Remark* 4.3.2. Since $\mathcal{X}$ is a balanced nilpotent section, $\mathcal{X}_{\mathcal{K}}$ is nilpotent. If the characteristic of $\mathcal{K}$ is zero, recall that by convention $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$.

*Proof.* We apply Proposition 4.2.4(a). According to that result, the condition $(\mathcal{X}_k)^{[p]} = 0$ shows that the weights of the image of $\phi_k$ on $\mathrm{Lie}(\mathcal{G}_k)$ are all $< 2p$. But then it is immediate that the weights of the image of $\phi_{\mathcal{K}}$ on $\mathrm{Lie}(\mathcal{G}_{\mathcal{K}})$ are all $< 2p$ and the same result shows that $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$. □

The following result is essentially a recapitulation of [McN03b, Theorem 13].

**Proposition 4.3.3.** *Assume that $\mathcal{G}$ is a* split *reductive group scheme over $\mathcal{A}$. If $(\mathcal{X}_k)^{[p]} = 0$ and if $\mathcal{X}_k$ is geometrically distinguished, there is a homomorphism of group schemes $\Psi : SL_{2,\mathcal{A}} \to \mathcal{G}$ such that $d\Psi(E_{\mathcal{A}}) = \mathcal{X}$ and $\Psi_{|S_{\mathcal{A}}} = \phi$.*

*Proof.* First of all, we may and will suppose that $\mathcal{G}$ is semisimple. Indeed, let $\mathcal{H} = \mathrm{der}(\mathcal{G})$ be the subgroup scheme of $\mathcal{G}$ described in [$SGA3_{III}$, Exp. XXII §6]. Then $\mathcal{H}$ is semisimple – see *loc. cit.*, Theorem 6.2.1. Moreover, $\mathcal{H}_k$ is the derived group of $\mathcal{G}_k$ and $\mathcal{H}_{\mathcal{K}}$ is the derived group of $\mathcal{G}_{\mathcal{K}}$; see *loc. cit.*, Remark 6.2.2. By assumption the cocharacter $\phi$ has the property that $\phi_{\mathcal{K}}$ is associated to $X_{\mathcal{K}}$ and $\phi_k$ is associated to $X_k$. It follows from the definition that $\phi_k$ takes values in $\mathcal{H}_k$ and $\phi_{\mathcal{K}}$ takes values in $\mathcal{H}_{\mathcal{K}}$. Thus we may and will replace $\mathcal{G}$ by $\mathcal{H}$ and thus suppose $\mathcal{G}$ to be semisimple.

Write $R = \mathbf{Z}_{(p)}$ and note that there is a unique unital ring homomorphism $R \to \mathcal{A}$. If $\mathcal{A}$ has "equal characteristic" – i.e. if $k$ and $\mathcal{K}$ have the same characteristic – the kernel of this homomorphism is $pR$; otherwise, this homomorphism is injective. Let $\mathcal{G}_R$ be a split semisimple group scheme over $R$ for which $\mathcal{G} = \mathcal{G}_R \times \mathrm{Spec}(\mathcal{A})$. There is a split maximal torus $\mathscr{T}_R$ of $\mathcal{G}_R$ such that $\mathscr{T}_R \times \mathrm{Spec}(\mathcal{A})$ identifies with $\mathscr{T}$. In particular, we may view $\phi$ also as a cocharacter of $\mathcal{G}_R$. As in Proposition 2.4.3, write $P \subset \mathcal{G}$ for the parabolic $\mathcal{A}$-subgroup scheme of $\mathcal{G}$ determined by $\phi$, and let $P_R \subset \mathcal{G}_R$ be the parabolic $R$-subgroup scheme of $\mathcal{G}_R$ determined by $\phi$. Note that $P = P_R \times \mathrm{Spec}(\mathcal{A})$. Moreover, write $U_R \subset P_R$ and $U \subset P$ for the smooth subgroup schemes of [$SGA3_{III}$, XXVI Prop. 1.21] whose fibers give the unipotent radicals of the fibers of $P_R$ resp. $P$.

Write $\mathrm{Lie}(\mathcal{G}) = \bigoplus_{n \in \mathbf{Z}} \mathrm{Lie}(\mathcal{G})(\phi; n)$ as a sum of weight spaces for the image of $\phi$. According to [McN05, Prop. 24], the condition that $(\mathcal{X}_k)^{[p]} = 0$ implies that $\mathrm{Lie}(\mathcal{G})(\phi; n) = 0$ when $n \geqslant 2p$. Hence also $\mathrm{Lie}(\mathcal{G}_R)(\phi; n) = 0$ when $n \geqslant 2p$. Now a second application of *loc. cit.* shows that the nilpotence class of the generic fiber $U_{\mathbf{Q}}$ of $U_R$ is $< p$.

Now [Sei00, Prop. 5.1] shows that the exponential isomorphism yields an isomorphism of $R$-group schemes $\exp : \mathrm{Lie}(U_R) \to U_R$. Hence by base change $R \to \mathcal{A}$, we get an exponential isomorphism $\exp : \mathrm{Lie}(U) \to U$. Applying these considerations to the opposite parabolic subgroup schemes, we have an exponential $\mathcal{A}$-isomorphism $\exp(\mathrm{Lie}(U^-)) \to U^-$.

Since $\mathcal{G}$ is semisimple, we may argue as in [McN03b, Lemma 10] to find a nilpotent element $\mathcal{Y} \in \operatorname{Lie}(\mathcal{G})(\phi;-2)$ such that $(\mathcal{X},\mathcal{Y},H = d\phi(1))$ is an $\mathfrak{sl}_2$-triple over $\mathcal{A}$. Indeed, since $\mathcal{X}_{\mathcal{K}}$ is distinguished, we have $\dim \operatorname{Lie}(\mathcal{G}_{\mathcal{K}})(\phi;0) = \dim \operatorname{Lie}(\mathcal{G}_{\mathcal{K}})(\phi;2)$. Moreover, Proposition 3.3.3 shows that that $C_{\mathcal{G}_k}(\mathcal{X}_k) \subset P(\phi_k)$. Thus $\operatorname{ad}(\mathcal{X}_k) : \operatorname{Lie}(\mathcal{G}_k)(\phi;-2) \to \operatorname{Lie}(\mathcal{G}_k)(\phi;0)$ is injective and is therefore a linear isomorphism. It follows at once that $\operatorname{ad}(\mathcal{X}) : \operatorname{Lie}(\mathcal{G})(\phi;-2) \to \operatorname{Lie}(\mathcal{G})(\phi;0)$ is an $\mathcal{A}$-isomorphism. In particular, there must be an element $\mathcal{Y} \in \operatorname{Lie}(\mathcal{G})(\phi;-2)$ with $\operatorname{ad}(\mathcal{X})(\mathcal{Y}) = [\mathcal{X},\mathcal{Y}] = H$, as required.

Let $\Psi_1 : \mathrm{SL}_{2,\mathcal{K}} \to \mathcal{G}_{\mathcal{K}}$ be the homomorphism of Theorem 4.2.2 with $d\Psi_1(E_1) = \mathcal{X}_{\mathcal{K}}$ and $\Psi_{1|S_1} = \phi_{\mathcal{K}}$. The "big cell" of $\mathrm{SL}_2$ is the $\mathcal{A}$-subscheme $\Omega = U_1^- S_1 U_1$ where $U_1^{\pm} \simeq \mathbf{G}_{a,\mathcal{A}}$ and $T_1 \simeq \mathbf{G}_{m,\mathcal{A}}$. The restriction of $\Psi_1$ to $\Omega_{\mathcal{K}}$ is then given by $(s,t,u) \mapsto \exp(s\mathcal{Y})\phi_{\mathcal{K}}(t)\exp(u\mathcal{X})$; thus $\Psi_{1|\Omega_{\mathcal{K}}}$ arises by base change from an $\mathcal{A}$-morphism $\Omega \to \mathcal{G}$. Now the argument found in the proof of [Ser96, Prop. 2] yields the required $\mathcal{A}$-homomorphism $\Psi : \mathrm{SL}_2 \to \mathcal{G}$. (The argument of *loc. cit.* uses that $\mathrm{SL}_{2,\mathbf{Z}}$ is covered by the big cell $\Omega_{\mathbf{Z}}$ together with $w\Omega_{\mathbf{Z}}$ for a suitable $w \in \mathrm{SL}_2(\mathbf{Z})$). □

*Remark* 4.3.4. As was already pointed out, the preceding result is essentially that found in [McN03b, Theorem 13], which in turn used the construction(s) found in [Ser96]. In fact, the underlying argumentation give here is essentially that of the proof of Theorem 4.2.2 given in [McN05].

Our remaining goal for this section is to extend the result Proposition 4.3.3 to cover the case when $\mathcal{G}$ is no longer assumed to be split. Here is our main tool:

**Proposition 4.3.5.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be affine $\mathcal{A}$-schemes which are flat and of of finite type over $\mathcal{A}$. Suppose that the DVR $\mathcal{B}$ is an étale extension of $\mathcal{A}$, and write $\mathcal{L}$ for the field of fractions of $\mathcal{B}$.*

*Let $\Psi : \mathcal{X}_{\mathcal{B}} \to \mathcal{Y}_{\mathcal{B}}$ be a $\mathcal{B}$-morphism, and suppose that the $\mathcal{L}$-morphism $\Psi_{\mathcal{L}} : \mathcal{X}_{\mathcal{L}} \to \mathcal{Y}_{\mathcal{L}}$ obtained from $\Psi$ arises by base change from a $\mathcal{K}$-morphism $\Psi_1 : \mathcal{X}_{\mathcal{K}} \to \mathcal{Y}_{\mathcal{K}}$. Then there is a unique $\mathcal{A}$-morphism $\Phi : \mathcal{X} \to \mathcal{Y}$ for which $\Psi = \Phi_{\mathcal{B}}$.*

*Proof.* After possibly replacing $\mathcal{B}$ – and hence also $\mathcal{L}$ – by a larger unramified extension, we may and will suppose that the residue field $\ell$ of $\mathcal{B}$ is a Galois extension of the residue field $k$ of $\mathcal{A}$. Write $\Gamma = \operatorname{Gal}(\ell/k)$; since $\mathcal{L}$ is unramified over $\mathcal{K}$, also $\mathcal{L}$ is a Galois extension of $\mathcal{K}$, and $\operatorname{Gal}(\mathcal{L}/\mathcal{K})$ identifies with $\Gamma$.

Since $\mathcal{X}$ is flat over $\mathcal{A}$, $\mathcal{A}[\mathcal{X}]$ identifies with an $\mathcal{A}$-subalgebra of $\mathcal{K}[\mathcal{X}_{\mathcal{K}}]$. Similarly, $\mathcal{B}[\mathcal{X}_{\mathcal{B}}]$ identifies with a $\mathcal{B}$-subalgebra of $\mathcal{L}[\mathcal{X}_{\mathcal{L}}]$. Moreover, $\mathcal{K}[\mathcal{H}_{\mathcal{K}}] = \mathcal{L}[\mathcal{X}_{\mathcal{L}}]^{\Gamma}$ and $\mathcal{A}[\mathcal{X}] = \mathcal{B}[\mathcal{X}_{\mathcal{B}}]^{\Gamma}$. Identical considerations hold for $\mathcal{Y}$.

Now suppose that $\Phi, \Phi' : \mathcal{X} \to \mathcal{Y}$ are $\mathcal{A}$-morphisms with $\Phi_{\mathcal{K}} = \Phi'_{\mathcal{K}}$; thus $\Phi$ and $\Phi'$ agree on the generic fiber $\mathcal{X}_\eta = \mathcal{X}_{\mathcal{K}}$. Since $\mathcal{X}$ is flat over , $\mathcal{X}_\eta$ is dense in $\mathcal{X}$ [Liu02, Lemma 4.3.7] so that $\Phi = \Phi'$; thus the uniqueness statement of the Proposition is immediate.

We now argue the required existence statement. Denote by $\Psi^* : \mathcal{B}[\mathcal{Y}] \to \mathcal{B}[\mathcal{X}]$ the comorphism of $\Psi : \mathcal{X} \to \mathcal{Y}$. Since by our assumption $\Psi_{\mathcal{L}}$ arises by base-change from a $\mathcal{K}$-mapping, it follows that $\Psi^*_{\mathcal{L}}$ is fixed by the action of $\Gamma$. But then $\Psi^*$ is already fixed by the action of $\Gamma$; thus $\Psi^*$ determines by restriction a mapping $\mathcal{A}[\mathcal{Y}] \to \mathcal{A}[\mathcal{X}]$ which is the comorphism of the desired mapping $\Phi$. □

We now state and prove the main result of this section: Recall that we have fixed a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ for $\mathcal{G}$.

**Theorem 4.3.6.** *If $\mathcal{X}_k^{[p]} = 0$, there is a unique $\mathcal{A}$-homomorphism*

$$\Phi : \mathrm{SL}_{2/\mathcal{A}} \to \mathcal{G}$$

*such that $d\Phi(E) = \mathcal{X}$, and $\Phi_{|\mathscr{D}} = \phi$.*

*Proof.* In view of Lemma 4.3.1 and our the assumption, we know that $(\mathcal{X}_k)^{[p]} = 0$ and $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$. Since $\phi_{\mathcal{K}}$ is a cocharacter associated with $\mathcal{X}_{\mathcal{K}}$, we may apply Theorem 4.2.2 to see that there is a unique $\mathcal{K}$-homomorphism $\psi_1 : \mathrm{SL}_{2,\mathcal{K}} \to \mathcal{G}_{\mathcal{K}}$ such that $d\psi_1(E_{\mathcal{K}}) = \mathcal{X}_{\mathcal{K}}$ and $\psi_{1|\mathscr{D}} = \phi_{\mathcal{K}}$, where as before $\mathscr{D}$ is the diagonal torus in $\mathrm{SL}_2$.

Provided that it exists, the uniqueness of $\Phi$ is now clear.

To argue the existence, we proceed as follows. According to Proposition 2.3.7, there is a DVR $\mathcal{B}$ which is a finite étale extension of $\mathcal{A}$ for which $\mathcal{G}_{\mathcal{B}}$ is a split reductive group scheme over $\mathcal{B}$. Now, Proposition 4.3.3 yields a $\mathcal{B}$-homomorphism $\Psi : \mathrm{SL}_{2,\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$ for which $d\Psi(E_{\mathcal{B}}) = \mathcal{X}_{\mathcal{B}}$ and $\Psi_{|\mathscr{D}} = \phi_{\mathcal{B}}$.

It follows from the uniqueness in Theorem 4.2.2 that $\Psi_{\mathcal{L}} = \psi_{1,\mathcal{L}}$. Thus Proposition 4.3.5 yields an $\mathcal{A}$-homomorphism $\Phi : \mathrm{SL}_2 \to \mathcal{G}$ for which $\Phi_{\mathcal{K}} = \Psi$. □

*Definition* 4.3.7. An $\mathcal{A}$-homomorphism $\Psi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{G}$ is said to be *optimal* provided that $\Psi_k$ and $\Psi_{\mathcal{K}}$ are optimal in the sense of Definition 4.2.3.

*Remark* 4.3.8. If $X \in \mathrm{Lie}(G)$ is a nilpotent element, recall Theorem 3.4.5 that we may find a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$ for $\mathcal{G}$ such that $\mathcal{X}_k = X$. Thus Theorem 1.7.1 is an immediate consequence of Theorem 4.3.6 together with the definition of a balanced triple.

## 5. Balanced sections for parahoric group schemes

Throughout this section, $G$ denotes a reductive group over $\mathcal{K}$ which splits over an unramified extension, $\mathcal{P}$ denotes a parahoric $\mathcal{A}$-group scheme for which $\mathcal{P}_{\mathcal{K}} = G$, and the residue characteristic $p = \mathrm{char}(k)$ satisfies the inequality $p > 2h - 2$ where $h = h_G$ denotes the Coxeter number of $G$.

Under these assumption, we may invoke Theorem 1.5.1; thus we may and will choose a subgroup scheme $\mathcal{M} \subset \mathcal{P}$ satisfying the conditions found in the statement of that Theorem. Now Proposition 2.1.5 guarantees that $G$ is a standard reductive group and $\mathcal{M}$ is standard reductive group scheme.

5.1. **Existence of balanced nilpotent sections for parahoric group schemes.** In this section, we are going to prove Theorem 1.7.2(a). We begin by formulating an "infinitesimal condition" that will permit us to recognize balanced nilpotent sections $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$.

Let $\mathcal{H}$ be a group scheme which is affine, smooth and of finite type over $\mathcal{A}$, and let $\mathcal{L}$ be an $\mathcal{H}$-module where $\mathcal{L}$ is free of finite rank as an $\mathcal{A}$-module.

**Proposition 5.1.1.** *Let* $x \in \mathcal{L}$. *Write* $\mathfrak{h} = \mathrm{Lie}(\mathcal{H})$, *and assume the following:*
*(a) the* $\mathcal{H}_{\mathcal{K}}$ *orbit of* $x_{\mathcal{K}}$ *is smooth – i.e.* $\dim \mathrm{Stab}_{\mathcal{H}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{h}_{\mathcal{K}}}(x_{\mathcal{K}})$, *and*
*(b)* $\dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{h}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim_k \mathfrak{c}_{\mathfrak{h}_k}(x_k)$.
*Then* $x$ *is balanced for the action of* $\mathcal{H}$.

*Proof.* Let $C = \mathrm{Stab}_{\mathcal{H}}(x)$. The group scheme $C_{\mathcal{K}}$ is smooth over $\mathcal{K}$ by assumption. It remains to argue that $C_k$ is smooth over $k$ and that $\dim C_{\mathcal{K}} = \dim C_k$.

It follows from Chevalley's upper semi-continuity theorem [EGAIV$_{III}$, §13.1.3] that

$$\dim C_{\mathcal{K}} \leqslant \dim C_k.$$

On the other hand, $\mathfrak{c}_{\mathfrak{g}_k}(x_k)$ is the Lie algebra of the group scheme $\mathrm{Stab}_{\mathcal{G}_k}(x_k) = C_k$. Thus $\dim C_k \leqslant \dim_k \mathfrak{c}_{\mathfrak{g}_k}(x_k)$ e.g. by [Knu+98, Lemma 21.8].

Combining these inequalities with our assumptions, we deduce that

$$\dim C_{\mathcal{K}} \leqslant \dim C_k \leqslant \dim_k \mathfrak{c}_{\mathfrak{g}_k}(x_k) = \dim_{\mathcal{K}} \mathfrak{c}_{\mathfrak{g}_{\mathcal{K}}}(x_{\mathcal{K}}) = \dim C_{\mathcal{K}}.$$

Thus equality holds everywhere, so indeed $C_k$ is smooth, e.g. by [Knu+98, Prop. 21.9], and moreover $\dim C_k = \dim C_{\mathcal{K}}$. ∎

Our main tool is the following result:

**Proposition 5.1.2.** *Let* $\Psi : \mathrm{SL}_{2,\mathcal{A}} \to \mathcal{M}$ *be a homomorphism which is* optimal *as in Definition 4.3.7.*
*(a)* $\mathcal{X} = d\Psi(E_{\mathcal{F}})$ *is a nilpotent section of* $\mathrm{Lie}(\mathcal{P})$ *which is balanced for the action of* $\mathcal{P}$, *and*
*(b)* $\mathrm{Lie}(\mathcal{P})$ *is a restricted* $\mathrm{SL}_{2,\mathcal{A}}$*-module via* $\Psi$.

*Proof.* Composing with the adjoint representation of $\mathcal{P}$ on $\mathrm{Lie}(\mathcal{P})$, observe that $\Psi_{\mathcal{K}}$ determines an $\mathrm{SL}_{2,\mathcal{K}}$-module structure on $\mathrm{Lie}(G) = \mathrm{Lie}(\mathcal{P}_{\mathcal{K}})$, $\Psi_k$ determines an $\mathrm{SL}_{2,k}$-module structure on $\mathrm{Lie}(\mathcal{P}_k)$ and $\Psi$ determines an $\mathrm{SL}_{2,\mathcal{A}}$-module structure on $\mathrm{Lie}(\mathcal{P})$.

It follows from Theorem A.1 that $\Psi_{\mathcal{K}}$ is an optimal homomorphism $\mathrm{SL}_{2,\mathcal{K}} \to G = \mathcal{G}_{\mathcal{K}}$. Since $p > 2h - 2$ it follows from Proposition 4.2.6 that $\mathrm{Lie}(G)$ is a restricted semisimple representation for $\mathrm{SL}_{2,\mathcal{K}}$ (for the module structure arising from $\Psi\mathcal{K}$ just mentioned), and that every simple $\mathrm{SL}_{2,\mathcal{K}}$-submodule has the form $L_{\mathcal{K}}(n)$ with $n \leqslant 2h - 2 \leqslant p - 1$.

It follows at once that the weights $m$ for the action of the torus $\mathscr{D}_{\mathcal{A}} \subset \mathrm{SL}_{2,\mathcal{A}}$ for which $\mathrm{Lie}(\mathcal{P})(m) \neq 0$ satisfy $|m| < p$; thus $\mathrm{Lie}(\mathcal{P})$ is a restricted module for $\mathrm{SL}_{2,\mathcal{A}}$ via $\Psi$. Now Proposition 4.1.7 implies that

$$\dim \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{X}_k) = \dim \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{X}_{\mathcal{K}}).$$

Finally, Proposition 5.1.1 implies that $\mathcal{X}$ is balanced for the action of $\mathcal{P}$, as required. ∎

We can now give the

*Proof of Theorem 1.7.2(a).* Recall that we are given a nilpotent element $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$.

First note that $M = \mathcal{M}_{\mathcal{K}}$ is an unramified reductive group, and the above discussion shows that the fibers of $\mathcal{M}$ are standard. Thus, we may invoke Theorem 1.6.1 to find a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ which is balanced for the action of $\mathcal{M}$ for which $\mathcal{X}_k = X_0$. We must argue that $\mathcal{X} \in \mathrm{Lie}(\mathcal{M}) \subset \mathrm{Lie}(\mathcal{P})$ is in fact balanced for the action of $\mathcal{P}$.

More precisely, we use Theorem 3.4.5 to find a *balanced triple* $(\mathcal{X}, \mathscr{S}, \phi)$ for $\mathcal{M}$ with $\mathcal{X}_k = X_0$. Since $p > 2h_M - 2$, Theorem 4.3.6 provides an $\mathcal{A}$-homomorphism $SL_{2,\mathcal{A}} \to \mathcal{M}$ with $d\Psi(E_{\mathcal{A}}) = \mathcal{X}$ which is optimal. Now Proposition 5.1.2 implies that $\mathcal{X}$ is balanced for the action of $\mathcal{P}$ and the proof is complete. □

5.2. **Conjugacy of balanced nilpotent sections.** In this section, we are going to prove the "conjugacy assertions" of the Theorems found in Section 1. More precisely, we are going to prove Theorem 1.6.1(c) and Theorem 1.7.2(b).

These conjugacy statements amount more-or-less to results obtained in [DeB02], specialized somewhat to the present setting. For the readers convenience, and for clarity, in this section we are going to recapitulate an adapted version of DeBacker's argument.

Throughout this section, $G$ will denote a connected and reductive algebraic group over the local field $\mathcal{K}$, and $\mathcal{P}$ will denote a parahoric group scheme over $\mathcal{A}$ with $\mathcal{P}_{\mathcal{K}} = G$.

We begin by recalling the *Moy-Prasad filtration* $\mathcal{P}(\mathcal{A})_s$ of the group $\mathcal{P}(\mathcal{A})$, indexed by real numbers $s \geqslant 0$. This filtration was introduced in [MP94]; see also [Adl98, §1.4]. We write $\mathcal{P}_+ \subset \mathcal{P}(\mathcal{A})$ for the subgroup which is the union the $\mathcal{P}(\mathcal{A})_s$ for $s > 0$.

Parallel to the Moy-Prasad filtration of $\mathcal{P}(\mathcal{A})$, there is an analogous Moy-Prasad filtration $\mathrm{Lie}(\mathcal{P})_s$ of the $\mathcal{A}$-Lie algebra $\mathrm{Lie}(\mathcal{P})$. We write $\mathrm{Lie}(\mathcal{P})_+$ for the union of the $\mathrm{Lie}(\mathcal{P})_s$ for $s > 0$. Then $\mathrm{Lie}(\mathcal{P})_+$ and each $\mathrm{Lie}(\mathcal{P})_s$ is a full $\mathcal{A}$-lattice in $\mathrm{Lie}(G)$.

Let $R = R_u(\mathcal{P}_k)$ denote the unipotent radical of the special fiber $\mathcal{P}_k$.

**Proposition 5.2.1.** *(a)* $\mathrm{Lie}(\mathcal{P}_+)$ *and* $\mathrm{Lie}(\mathcal{P})_s$ *for each* $s \geqslant 0$ *is a* $\mathcal{P}$*-submodule of* $\mathrm{Lie}(\mathcal{P})$ *for the adjoint action.*
*(b)* $\mathrm{Lie}(\mathcal{P})_+$ *is the kernel of the natural mapping* $\psi : \mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k) \to \mathrm{Lie}(\mathcal{P}_k/R)$.

*Proof.* (a) follows from [Adl98, Prop. 1.2.5]. As to (b), one may confirm the statement when $G$ is split using the definition of the Moy-Prasad filtration together with [BT84, Prop. 4.6.10] and [BT84, Cor. 4.6.7]. Since $G$ splits over an unramified extension, the parahoric group scheme $\mathcal{P}$ arises via étale descent from a a parahoric group scheme $\mathcal{Q}$ for $G_{\mathcal{L}}$ for some finite unramified extension $\mathcal{L}$ of $\mathcal{K}$. One knows that $\mathrm{Lie}(\mathcal{P})+ = \mathrm{Lie}(\mathcal{P}) \cap \mathrm{Lie}(\mathcal{Q})_+$ and the assertion follows immediately. □

The main tool of the present section is the following, which the reader may compare to results in [DeB02, §5.1]:

**Theorem 5.2.2.** *Let* $\mathcal{X}, \mathcal{X}' \in \mathrm{Lie}(\mathcal{P})$ *be* balanced *nilpotent sections for which* $\mathcal{X} + \mathcal{P}_+ = \mathcal{X}' + \mathcal{P}_+$*. Then there is an element* $g \in \mathcal{P}_+$ *for which* $\mathrm{Ad}(g)\mathcal{X} = \mathcal{X}'$*.*

We first remark that Theorem 1.6.1(c) and Theorem 1.7.2(b) are immediate consequences of Theorem 5.2.2. Before giving the proof of Theorem 5.2.2, we formulate some preliminary observations and auxiliary results.

Returning to the setting of Theorem 5.2.2, consider sections $\mathcal{X}, \mathcal{X}'$ as in the statement of that result; let us write $X_0 \in \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$ for the common image of $\mathcal{X}$ and $\mathcal{X}'$. Recall that we have chosen a subgroup scheme $\mathcal{M} \subset \mathcal{P}$ which is reductive over $\mathcal{A}$, for which $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$, and for which $\mathcal{M}_{\mathcal{K}}$ is a reductive subgroup of $G$ of type $C(\mu)$.

Since $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$, we may identify $X_0$ with a nilpotent element of $\mathrm{Lie}(\mathcal{M}_k)$; by some abuse of notation, we continue to denote this element by $X_0$.

According to Theorem 1.6.1 we may choose a balanced section $\mathcal{X}_1 \in \mathrm{Lie}(\mathcal{M})$ for which $\mathcal{X}_{1,k} = X_0$; according to Theorem 1.7.2, $\mathcal{X}_1$ is balanced for $\mathcal{P}$ when viewed as an element of $\mathrm{Lie}(\mathcal{P})$. It is evidently sufficient to prove Theorem 5.2.2 in the special case that $\mathcal{X} = \mathcal{X}_1$, which we now assume.

According to Theorem 1.7.1, we may find a homomorphism $\Phi : SL_2 \to \mathcal{M}$ such that $\Phi_{\mathcal{K}}$ is an optimal homomorphism for $\mathcal{X}_{\mathcal{K}}$ and $\Phi_k$ is an optimal homomorphism for $\mathcal{X}_k$. Since $p > 2h - 2$, proposition 4.2.6 shows that $\mathrm{Lie}(\mathcal{P})$ is a restricted representation for $SL_{2,\mathcal{A}}$ via $\mathrm{Ad} \circ \Phi$.

Recall the notation $E, F$ for the "standard" nilpotent sections of $\mathrm{Lie}(SL_{2,\mathcal{A}})$, where $d\Phi(E) = \mathcal{X}$. Let $S \subset \mathrm{Lie}(G)$ denote the kernel of the operator $\mathrm{ad}\, d\Phi(F)$. For each $s \geqslant 0$, note that $S_s = S \cap \mathrm{Lie}(\mathcal{P})_s$ is of course the kernel of the restriction of $\mathrm{ad}\, d\Phi(F)$ to $\mathrm{Lie}(\mathcal{P})_s$, and $S_{s+} = S \cap \mathrm{Lie}(\mathcal{P})_{s+}$ is the kernel of that operator on $\mathrm{Lie}(\mathcal{P})_{s+}$.

**Proposition 5.2.3.** *For each* $s \geqslant 0$, *we have* $\mathrm{Lie}(\mathcal{P})_s = S_s + [\mathcal{X}, \mathrm{Lie}(\mathcal{P})_s]$

*Proof.* Indeed, since $p > 2h-2$ we have observed already that $\mathrm{Lie}(\mathcal{P})$ is a restricted module for $SL_{2,\mathcal{A}}$. It is clear from definitions that any $SL_{2,\mathcal{A}}$-submodule of a restricted module is itself restricted. Thus by Proposition 5.2.1, $\mathrm{Lie}(\mathcal{P})_s$ is a restricted module for $SL_{2,\mathcal{A}}$. Now, Proposition 4.1.7 shows that $\mathrm{Lie}(\mathcal{P})_s$ is the internal direct sum of $\ker(F)$ and $[E, \mathrm{Lie}(\mathcal{P})_s]$. But $S_s$ is the kernel of the action of $F$ on $\mathrm{Lie}(\mathcal{P})_s$ and $E$ acts via $\mathcal{X}$, and the Proposition follows at once. □

**Proposition 5.2.4.**

$$\mathrm{Ad}(P_+)(X + S_+) = X + \mathrm{Lie}(\mathcal{P})_+.$$

*Proof.* This amounts to the $r = 0$ case of [DeB02, Lemma 5.2.1]. This result is valid under our hypotheses; indeed, the proof give in *loc. cit.* uses the existence of "mock exponential" found in [Adl98] together with Proposition 5.2.3 [3]. □

**Proposition 5.2.5.** *Let* $\mathcal{F}$ *be an algebraically closed field of characteristic* $p$, *let* $H$ *be a reductive group over* $\mathcal{F}$ *and suppose that either* $p = 0$ *or* $p > 2h_H - 2$. *Let* $X \in \mathrm{Lie}(H)$ *be nilpotent, and let* $\Phi : SL_2 \to H$ *be an optimal homomorphism determined by* $X$. *If* $S = \mathfrak{c}_{\mathrm{Lie}\, H}\, d\Phi(F)$, *then* $\mathrm{Ad}(G)X \cap (X + S) = \{X\}$.

*Proof.* Write $Y$ for the $G$-orbit of $X$. Since the centralizer of $X$ in $G$ is smooth section 2.1, the tangent space of $Y$ at $X$ is $T_X Y = [X, \mathrm{Lie}(G)]$. Moreover, $S = T_X(X + S)$.

It follows that the subvarieties $Y$ and $X + S$ are transversal at $X$, and in particular, there is an open subvariety $U \subset S$ such that $X \in U(\mathcal{F})$ and such that $(X + U) \cap Y = \{X\}$.

Now, it follows from [Jan04, §7.15] that the weights of the image of $\phi$ on $S$ are *strictly negative*. Precisely as in the proof of [CG10, Prop. 3.7.15], the action of the image of $\phi$ determines an action of $\mathbf{G}_m$ on $X + S$ for which $X$ is the only fixed point, and as in *loc. cit.* it follows that $(X + S) \cap Y = \{X\}$. □

*Proof of Theorem 5.2.2.* In view of Theorem 3.4.5, it suffices to prove the result when $\mathcal{X}$ appears in a balanced triple $(\mathcal{X}, \mathscr{S}, \phi)$.

Since $\mathcal{X}_k = \mathcal{Y}_k$, we have $\mathcal{Y} = \mathcal{X} + \pi\mathcal{Z}$ for some $\mathcal{Z} \in \mathrm{Lie}(\mathcal{G})$. Now according to Proposition 5.2.4, there is an element $g \in \mathcal{P}_+$ and an element $W \in S$ such that

$$\mathrm{Ad}(g)(\mathcal{X} + \pi W) = \mathcal{X} + \pi\mathcal{Z} = \mathcal{Y}.$$

Since $\mathcal{Y}_k = \mathcal{X}_k$ and since $\mathcal{X}$ and $\mathcal{Y}$ are balanced, the dimension of the orbit of $\mathcal{Y}_\mathcal{K}$ coincides with that of the orbit of $\mathcal{X}_\mathcal{K}$. It now follows from Proposition 5.2.5 – applied to an algebraic closure of $\mathcal{K}$ – that $W = 0$; thus indeed $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{G}(\mathcal{A})$-conjugate. □

## 6. Conclusions

Throughout this section, $G$ denotes a connected and reductive group over the local field $\mathcal{K}$. We suppose that $G$ splits over an unramified extension of $\mathcal{K}$, and that $p > 2h-2$. In particular, $G$ is a standard reductive group.

### 6.1. **Completion of the proofs.**

In this section, we give the proof for Theorem 1.7.3 and the proof of the main result, Theorem 1.3.1.

Recall that in the statement of Theorem 1.7.3, $X_1 \in \mathrm{Lie}(G)$ is a $\mathcal{K}$-rational nilpotent element; we must find a parahoric group scheme $\mathcal{P}$ and a balanced section $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ for which $X_1 = \mathcal{X}_\mathcal{K}$. We give here an adaptation of the argument of [DeB02, Lemma 4.5.3]; Debacker attributes this argument to Gopal Prasad.

We first establish the following preliminary result:

**Proposition 6.1.1.** *Let* $X_1 \in \mathrm{Lie}(G)$ *be nilpotent, and let* $\phi_1$ *be a cocharacter of* $G$ *associated with* $X_1$. *Write* $p$ *for the characteristic of* $\mathcal{K}$. *If* $(X_1)^{[p]} = 0$ [4], *then there is a parahoric group scheme* $\mathcal{P}$ *with generic fiber* $\mathcal{P}_\mathcal{K} = G$ *and an* $\mathcal{A}$*-homomorphism* $\Phi : SL_{2,\mathcal{A}} \to \mathcal{P}$ *with the following properties:*

[3] The argument we require from [DeB02] begins on p. 24 following displayed equation (6).

[4] Recall that if the characteristic of $\mathcal{K}$ is zero, we simply define $(X_1)^{[p]} = 0$.

*(a) if* $\phi = \Phi_{|\mathscr{D}}$, *then* $\phi_{\mathcal{K}} = \phi_1$, *and*
*(b) if* $\mathcal{X} = d\Phi(E)$, *then* $\mathcal{X}_{\mathcal{K}} = X_1$.

*Proof.* It suffices to give the proof after replacing G by its derived group. Indeed, let $G_1$ denote the derived group of G. Fix a maximal split torus $T_1$ of $G_1$ contained in a maximal split torus T of G, and write $Z^0$ for the split component of the center of G (thus $Z^0$ is a split torus over $\mathcal{K}$). The product mapping $Z^0 \times T_1 \to T$ is an isogeny, and hence $X_*(T) \otimes \mathbf{Q}$ is the direct sum of $X_*(T_1) \otimes \mathbf{Q}$ and $X_*(Z^0) \otimes \mathbf{Q}$; write

$$\pi : S = X_*(T) \otimes \mathbf{Q} \to S_1 = X_*(T_1) \otimes \mathbf{Q}$$

for the natural projection map. For $x \in S$, write $\mathcal{P}_x$ for the parahoric subgroup scheme with generic fiber G determined by x and write $\mathcal{Q}_{\pi(x)}$ for the parahoric group scheme with generic fiber $G_1$ determined by $\pi(x)$.

It is clear from the construction that

$$\mathcal{P}_x \supset \mathcal{Q}_{\pi(x)} \quad \text{and} \quad \mathcal{P}_x = \mathcal{P}_{\pi(x)}.$$

So the result will follow if we construct a homomorphism $SL_{2,\mathcal{A}} \to \mathcal{Q}_{\pi(x)}$ with properties (a) and (b). Thus we may and will suppose G to be *semisimple* for the remainder of the proof.

Let $\Phi_1 : SL_{2,\mathcal{K}} \to G = \mathcal{G}_{\mathcal{K}}$ be the optimal homomorphism determined by X and $\phi_1$ as in Theorem 4.2.2. We now argue as in [DeB02, Lemma 4.5.3]. Thus, we write $\mathcal{K}_u$ for the maximal unramified extension of $\mathcal{K}$, we consider the Bruhat-Tits building $\mathcal{B}_u$ of $G_{\mathcal{K}_u}$, and we write $\Gamma = \mathrm{Gal}(\mathcal{K}_u/\mathcal{K}) = \mathrm{Gal}(\overline{k}/k)$.

Let $\mathcal{A}_u$ denote the integral closure of $\mathcal{A}$ in $\mathcal{K}_u$ and consider the action of $SL_2(\mathcal{A}_u) \times \Gamma$ on $\mathcal{B}_u$, where the action of $SL_2(\mathcal{A}_u)$ is determined by $\Phi_{1,\mathcal{K}_u}$. Since the group $SL_2(\mathcal{A}_u) \times \Gamma$ is bounded, [Tit79, §2.3.1] shows that there is a fixed point $x \in \mathcal{B}_u$ for this action.

Since G is semisimple, [BT84, §4.6.27] shows that the stabilizer in $G(\mathcal{K}_u)$ of x is the group of $\mathcal{A}_u$ points of a smooth group scheme $\mathcal{H}$ with generic fiber G, and the parahoric $\mathcal{A}_u$-group scheme $\mathcal{Q} = \mathcal{Q}_x$ with generic fiber $G_{\mathcal{K}_u}$ is an $\mathcal{A}_u$ subgroup scheme of $\mathcal{H}$ which is equal to the identity component of $\mathcal{H}$. Since x is stable by the Galois group $\Gamma$, it follows that x is actually in the Bruhat-Tits building of G. In particular, there is a parahoric $\mathcal{A}$-group scheme $\mathcal{P} = \mathcal{P}_x$ with generic fiber G for which $\mathcal{Q} = \mathcal{P}_{\mathcal{A}_u}$.

Since x is stable by $SL_2(\mathcal{A}_u)$, it follows that $\Phi_1$ maps $SL_2(\mathcal{A}_u)$ to the group of points $\mathcal{H}(\mathcal{A}_u)$. Now, the group scheme $SL_2$ is *étoffé* [BT84, §1.7], hence it follows from [BT84, (1.7.1)] that there is an $\mathcal{A}_u$-homomorphism $\Psi : SL_{2,\mathcal{A}_u} \to \mathcal{H}$ compatible with the mapping $\Phi_1$ on $\mathcal{A}_u$-points.

Since $SL_2$ has connected fibers, it follows that $\Psi$ factors through the subgroup scheme $\mathcal{Q} \subset \mathcal{H}$; i.e. $\Psi$ determines a homomorphism $\Psi : SL_{2,\mathcal{A}_u} \to \mathcal{Q} = \mathcal{P}_{\mathcal{A}_u}$.

Finally, étale descent shows that $\Psi$ arises by base change from an $\mathcal{A}$-homomorphism $\Psi_0 : SL_{2,\mathcal{A}} \to \mathcal{P}$. It is immediate that $\Psi_0$ satisfies conditions (a) and (b). □

We now give:

*Proof of Theorem 1.7.3:* According to Proposition 4.2.4, the assumption $p > 2h-2$ implies that $(\mathcal{X}_{\mathcal{K}})^{[p]} = 0$. Thus we may apply Proposition 6.1.1 to find an $\mathcal{A}$-homomorphism $\Phi : SL_{2,\mathcal{A}} \to \mathcal{P}$ as in the statement of that Proposition.

We write $\psi : \mathbf{G}_{m,\mathcal{A}} \to \mathcal{P}$ for the $\mathcal{A}$-homomorphism obtained by restricting $\Phi_1$ to the diagonal torus of $SL_{2,\mathcal{A}}$. Since $p > 2h-2$, Proposition 4.2.4 shows that $\mathrm{Lie}(G)(\psi_{\mathcal{K}}; i) \neq 0 \implies |i| < p$ and hence that $\mathrm{Lie}(\mathcal{P})(\psi; i) \neq 0 \implies |i| < p$. It follows that $\mathrm{Lie}(\mathcal{P})$ is a restricted module for $SL_{2,\mathcal{A}}$. We now argue as in the proof Proposition 5.1.2; it follows from Proposition 4.1.7 that $\mathcal{X} = d\Phi(E_{\mathcal{A}})$ is balanced for the action of $\mathcal{P}$, as required. □

Finally, we give the proof of the main result of this paper:

*Proof of Theorem 1.3.1:* We are given a nilpotent element $X_1 \in \mathrm{Lie}(G)$ and we seek a reductive subgroup M of G satisfying the conditions given in the statement of the Theorem.

Since $C_G(X_1)$ is smooth over $\mathcal{K}$, we may invoke Proposition 2.3.9 to find a maximal unramified $\mathcal{K}$-torus $T \subset C_G(X_1)$. The centralizer $H = C_G(T)$ is a reductive group over $\mathcal{K}$ which splits over an unramified extension of $\mathcal{K}$. If we find a subgroup M of H satisfying the conclusion of Theorem 1.3.1, the Theorem will follows. Thus, we may and will suppose: $(\diamondsuit)$ an unramified torus of $C_G(X_1)$ is central in G.

We now use Theorem 1.7.3 to find data $\mathcal{X}, \mathcal{P}, \mathcal{M}$ as in the statement of that Theorem. We then know the following:

(i) $\mathcal{P}$ is a parahoric group scheme for the group G,
(ii) $\mathcal{M}$ is a reductive $\mathcal{A}$-subgroup scheme of $\mathcal{P}$,
(iii) $\mathcal{X} \in \mathrm{Lie}(\mathcal{M})$ is a balanced nilpotent section with $\mathcal{X}_{\mathcal{K}} = X_1$,
(iv) $\mathcal{M}_{\mathcal{K}}$ is a reductive subgroup of G of type $C(\mu)$.

It is now clear that $X_1 \in \mathrm{Lie}(\mathcal{M}_{\mathcal{K}})$, so it only remains to confirm that $X_1$ is geometrically distinguished for the action of $\mathcal{M}_{\mathcal{K}}$. Now use Proposition 3.5.2 to find an $\mathcal{A}$-torus $\mathscr{T}$ of $\mathcal{M}$ centralizing $\mathcal{X}$ for which $\mathscr{T}_{\mathcal{F}}$ is a maximal torus of $C_{\mathcal{F}}$ for $\mathcal{F} = k, \mathcal{K}$, where $C = C_{\mathcal{M}}(\mathcal{X})$.

The unramified torus $\mathscr{T}_{\mathcal{K}}$ in $G = \mathcal{G}_{\mathcal{K}}$ centralizes $X_1 = \mathcal{X}_{\mathcal{K}}$, so $(\diamondsuit)$ implies that $\mathscr{T}_{\mathcal{K}}$ is a central in G, and hence central in $\mathcal{M}_{\mathcal{K}}$. Since $\mathscr{T}_{\mathcal{K}}$ is a maximal torus of $C_{\mathcal{K}}$, this confirms that $X_1$ is geometrically distinguished in $M = \mathcal{M}_{\mathcal{K}}$ and completes the proof of Theorem 1.3.1. □

6.2. **DeBacker's parametrization.** This section investigates the relationship between the balanced nilpotent sections of a parahoric group scheme with the parametrization of nilpotent orbits given by Debacker in [DeB02].

As before, G denotes a reductive group over $\mathcal{K}$ which splits over an unramified extension of $\mathcal{K}$, and we suppose that $p > 2h-2$. We are going to describe DeBacker's parametrization for any parahoric group scheme $\mathcal{P}$ for G, which we now fix.

Recall that according to Proposition 5.2.1, the subalgebra $\mathrm{Lie}(\mathcal{P})_+ \subset \mathrm{Lie}(\mathcal{P})$ determined by the Moy-Prasad filtration coincides with the kernel of the natural mapping

$$\mathrm{Lie}(\mathcal{P}) \to \mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k).$$

We are going to prove:

**Theorem 6.2.1.** *Let $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ be a balanced nilpotent section.*

*(a) For any $\mathcal{Y} \in \mathcal{X} + \mathrm{Lie}(\mathcal{P})_+$, we have $\dim \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{Y}_{\mathcal{K}}) \leqslant \dim \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{X}_{\mathcal{K}})$.*
*(b) The $G(\mathcal{K})$-orbit of $\mathcal{X}_{\mathcal{K}}$ is the rational nilpotent orbit of smallest dimension which intersects the coset $\mathcal{X} + \mathrm{Lie}(\mathcal{P})_+$ non-trivially.*

*Remark* 6.2.2. To achieve the parametrization of nilpotent orbits in the "depth zero" ($r = 0$) case in [DeB02], DeBacker assigns to a nilpotent orbit $\mathcal{P}(k)$-orbit in $\mathrm{Lie}(\mathcal{P})/\mathrm{Lie}(\mathcal{P})_+$ the nilpotent $G(\mathcal{K})$-orbit of smallest dimension intersecting the $\mathrm{Lie}(\mathcal{P})_+$. Thus, Theorem 6.2.1 confirms that the balanced nilpotent sections describe DeBacker's assignment.

Before giving the proof of Theorem 6.2.1, we begin with some observations about adjoint conjugacy classes in linear algebraic groups having a Levi decomposition.

Let H be a linear algebraic group over a field $\mathcal{F}$. Suppose that the unipotent radical $R = R_uH$ is defined over $\mathcal{F}$ and that H has a Levi decomposition – i.e. there is a closed $\mathcal{F}$-subgroup $M \subset H$ such that the restriction of the quotient mapping $\pi : H \to H/R$ induces an isomorphism $\pi_{|M} : M \simeq H/R$. Let $X_0 \in \mathrm{Lie}(H/R)$ and let $X \in \mathrm{Lie}(M)$ be the unique element with $d\pi(X) = X_0$.

Choose a filtration

$$(\clubsuit) \quad \mathrm{Lie}(H) = L^0 \supset L^1 \supset L^2 \cdots \supset L^d = 0$$

of $\mathrm{Lie}(H)$ by H-invariant subspaces for which R acts trivially on the quotient $L^i/L^{i+1}$ for each $0 \leqslant i \leqslant d-1$; such a filtration exists since the unipotence of R means that $V^R \neq \{0\}$ for any R-module $V \neq \{0\}$.

**Proposition 6.2.3.** *For each $Y \in d\pi^{-1}(X_0)$, we have the inequality*

$$\sum_{i=0}^{d} \ker(\overline{\mathrm{ad}(X)} : L^i/L^{i+1} \to L^i/L^{i+1}) \geqslant \dim \mathfrak{c}_{\mathrm{Lie}(H)} Y,$$

*where $\overline{\mathrm{ad}(X)}$ is the endomorphism of $L^i/L^{i+1}$ induced by $\mathrm{ad}(X)$ (for each $i$).*

*Proof.* Let $Y \in d\pi^{-1}(X_0)$, and notice that $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ induce the same mapping on the associated graded space $\mathrm{gr}(\mathrm{Lie}(H)) = \bigoplus_{i=0}^{d-1} L^i/L^{i+1}$.

Now the Proposition follows from the observation that the dimension of the kernel of the mapping $\mathrm{gr}(\mathrm{ad}(Y))$ exceeds the dimension of $\ker(\mathrm{ad}(Y)) = \mathfrak{c}_{\mathrm{Lie}(H)}(Y)$. □

**Corollary 6.2.4.** *Suppose that* $\Phi : SL_{2,\mathcal{F}} \to M$ *is an* $\mathcal{F}$*-homomorphism, and that – in the notation of section 4.2 – we have* $X = d\phi(E_1)$*. If* $\mathrm{Lie}(H)$ *is a completely reducible* $SL_{2,\mathcal{F}}$*-module, then*

$$\dim \mathfrak{c}_{\mathrm{Lie}(H)}(X) \geqslant \dim \mathfrak{c}_{\mathrm{Lie}(H)}(Y)$$

*for all* $Y \in d\pi^{-1}(X_0)$*.*

*Proof.* Choose a filtration (♣) as above.

The assumption of complete reducibility implies that – as modules for $SL_{2,\mathcal{F}}$ –

$$\mathrm{Lie}(H) \simeq \bigoplus_{i=0}^{d-1} L^i/L^{i+1}.$$

Thus the dimension of the centralizer in $\mathrm{Lie}(H)$ of $X = d\phi(E_1)$ is precisely

$$\sum_{i=0}^{d-1} \dim \ker(\overline{\mathrm{ad}\, X} : L^i/L^{i+1} \to L^i/L^{i+1});$$

the result now follows from Proposition 6.2.3. □

*Proof of Theorem 6.2.1.* Fix a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in Theorem 1.5.1. Since $\mathcal{M}_k$ is a Levi factor of $\mathcal{P}_k$, we may choose $X_0 \in \mathrm{Lie}(\mathcal{M}_k)$ which identifies with the image of $\mathcal{X}_k$ in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$.

Now, $\mathcal{M}$ is a reductive group scheme. If $h_M$ denotes the Coxter number of $\mathcal{M}_\mathcal{K}$, then we have $p > 2h - 2 \geqslant 2h_M - 2$ by Proposition 2.2.1. In particular, $\mathcal{M}$ is a standard reductive group scheme over $\mathcal{A}$.

Thus, we may use Theorem 1.6.1 to find a section $\mathcal{X}_0 \in \mathrm{Lie}(\mathcal{M})$ which is balanced for $\mathcal{M}$ with $X_0 = \mathcal{X}_{0,k}$. We may also invoke Theorem 1.7.1 to find an $\mathcal{A}$-homomorphism $\Phi : SL_{2/\mathcal{A}} \to \mathcal{M}$ such that $\Phi_k$ is optimal for $\mathcal{X}_k$ and $\Phi_\mathcal{K}$ is optimal for $\mathcal{X}_\mathcal{K}$. Finally, the representations

Now, Theorem 1.7.2 guarantees that $\mathcal{X}_0 \in \mathrm{Lie}(\mathcal{M})$ is balanced for $\mathcal{P}$. Finally, since $\mathcal{X}$ and $\mathcal{X}_0$ have the same image in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k)$, Theorem 1.7.2(c) shows that $\mathcal{X}$ and $\mathcal{X}_0$ are conjugate by an element of $\mathcal{P}(\mathcal{A})$.

Thus it suffices to give the proof when $\mathcal{X} = \mathcal{X}_0 \in \mathrm{Lie}(\mathcal{M})$. We now prove (a). Recall that $\mathcal{Y} \in \mathcal{X} + \mathrm{Lie}(\mathcal{P})_+$. First, Chevalley's upper semi-continuity theorem [EGAIV$_{III}$, §13.1.3] implies that

$$(\flat) \quad \dim_\mathcal{K} \mathfrak{c}_{\mathrm{Lie}(G)}(\mathcal{Y}_\mathcal{K}) \leqslant \dim \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{Y}_k).$$

Now Corollary 6.2.4 shows that

$$(*) \quad \dim_k \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{Y}_k) \leqslant \dim_k \mathfrak{c}_{\mathrm{Lie}(\mathcal{P}_k)}(\mathcal{X}_k).$$

Combining $(\flat)$ and $(*)$ and using the fact that $\mathcal{X} \in \mathrm{Lie}(\mathcal{P})$ is balanced, one deduces assertion (a).

To prove (b), suppose that $\mathcal{Y} = \mathcal{X} + \mathcal{Z}$ for $\mathcal{Z} \in \mathrm{Lie}(\mathcal{P})_+$ and that equality holds for the inequality in (a). Since G is a geometrically standard reductive group, the identity component of $C_G(\mathcal{Y}_\mathcal{K})$ is a smooth group scheme. Thus Proposition 5.1.1 implies that $\mathcal{Y}$ is a balanced nilpotent section. Since the image of $\mathcal{Y}$ in $\mathrm{Lie}(\mathcal{P}_k/R_u\mathcal{P}_k) = \mathrm{Lie}(\mathcal{P})/\mathrm{Lie}(\mathcal{P})_+$ coincides with that of $\mathcal{X}$, it follows from Theorem 1.7.2(b) that $\mathcal{X}$ and $\mathcal{Y}$ are conjugate by an element of $\mathcal{P}(\mathcal{A})$, and the result follows. □

6.3. **Examples.** We keep the notations $\mathcal{K}, \mathcal{A}, k$ from before. We suppose that the characteristic of the residue field $k$ satisfies $p > 2$.

Consider a $\mathcal{K}$-vector space V together with a non-degenerate bilinear form $\beta$, and assume that $\beta$ is either symmetric or alternating. Then $\beta$ determines an involution $\iota = \iota_\beta = (X \mapsto X^*)$ of the simple $\mathcal{K}$-algebra $B = \mathrm{End}_\mathcal{K}(V)$ by the rule $\beta(Xv, w) = \beta(v, X^*w)$ for each $X \in B$ and all $v, w \in V$, and in turn the algebra-with-involution $(B, \iota)$ determines a simple algebraic $\mathcal{K}$-group $\mathrm{Iso}(B, \iota)$ [Knu+98, §23]. When $\beta$ is alternating, $\mathrm{Iso}(B, \iota) = \mathrm{Sp}(V) = \mathrm{Sp}(V, \beta)$ is the split symplectic group, and when $\beta$ is symmetric, $\mathrm{Iso}(B, \iota) = O(V, \beta)$ is the orthogonal group determined by $\beta$. Recall that $O(V, \beta)$ is $\mathcal{K}$*-split* if and only if $\beta$ is *hyperbolic*.

The Lie algebra of $G = \mathrm{Iso}(A, \iota)$ may be described by

$$\mathrm{Lie}(G) = \mathrm{Skew}(B, \iota) = \{X \in B \mid X + X^* = 0\}.$$

We will refer to the pair $(V, \beta)$ as a *symplectic space*, resp. an *orthogonal space*, if $\beta$ is alternating, resp. symmetric. Let us fix an orthogonal space $(W, \gamma)$ with $\dim W = m$.

**Observation 6.3.1.** *(a) If* $(V_1, \beta_1)$ *is a symplectic space, then* $(V_1 \otimes W, \beta_1 \otimes \gamma)$ *is a symplectic space.*

*(b) If* $(V_1, \beta_1)$ *is a hyperbolic orthogonal space with* $\dim V \geqslant 2$*, then* $(V_1 \otimes W, \beta_1 \otimes \gamma)$ *is a hyperbolic orthogonal space.*

We consider two possibilities for the pair $(V, \beta)$. In the first case, suppose that $\beta$ is alternating and that $\dim V = 2nm$. Then there is an isomorphism of symplectic spaces $(V, \beta) \simeq (V_1 \otimes W, \beta_1 \otimes \gamma)$ where $(V_1, \beta_1)$ is a symplectic space of dimension $2n$.

In the second case, $\beta$ is symmetric and hyperbolic, and $\dim V = nm$. Then there is an isomorphism of orthogonal spaces $(V, \beta) \simeq (V_1 \otimes W, \beta_1 \otimes \gamma)$ where $(V_1, \beta_1)$ is a hyperbolic orthogonal space of dimension $n$.

In both cases, we consider the algebras $B = \mathrm{End}_{\mathcal{K}}(V)$, $B_1 = \mathrm{End}_{\mathcal{K}}(V_1)$ and $C = \mathrm{End}_{\mathcal{K}}(W)$ with respective involutions $\iota = \iota_\beta, \iota_1 = \iota_{\beta_1}, j = \iota_\gamma$, and the corresponding $\mathcal{K}$-groups

$$G = \mathrm{Iso}(B, \iota), G_1 = \mathrm{Iso}(B_1, \iota_1), H = \mathrm{Iso}(C, j).$$

**Observation 6.3.2.** $B_1 \otimes_{\mathcal{K}} C$ *identifies with an* $\iota$*-invariant subalgebra of* $B$*, and the restriction of* $\iota$ *to this subalgebra identifies with the involution* $\iota_1 \otimes j$*. In particular, there is a* $\mathcal{K}$*-homomorphism* $\phi : G_1 \times H \to G$ *with central kernel.*

**Observation 6.3.3.** *There is a* regular nilpotent *element* $X_1 \in \mathrm{Lie}(G_1) = \mathrm{Skew}(B_1, \iota_1)$*. Write* $X$ *for the nilpotent element* $X_1 \otimes 1_C \in \mathrm{Skew}(B_1 \otimes C, \iota_1 \otimes j) \subset \mathrm{Skew}(B, \iota) = \mathrm{Lie}(G)$*. The reductive quotient of the centralizer of* $X$ *identifies with* $H = O(W, \gamma)$*.*

For the remainder of this discussion, let us fix a nilpotent element $X$ as in the preceding observation. Then the centralizer of $X$ has an unramified maximal $\mathcal{K}$-torus if and only if $O(W, \gamma)$ has an unramified maximal $\mathcal{K}$-torus.

Now suppose that $\dim W = 2$ and that $(W, \gamma)$ identifies with $(\mathcal{L}, N_{\mathcal{L}/\mathcal{K}})$ where $\mathcal{L}$ is a quadratic separable extension of $\mathcal{K}$ and where $N_{\mathcal{L}/\mathcal{K}}$ denotes the norm form.

**Observation 6.3.4.** *The identity component* $SO(W, \beta) = O^o(W, \beta)$ *identifies with the 1-dimensional* $\mathcal{K}$*-torus* $\mathbf{G}^1_{m,\mathcal{L}}$ *(the "norm-one-torus").*

In particular, when $\mathcal{K} \subset \mathcal{L}$ is a ramified quadratic extension, the torus $SO(W, \beta)$ is not an unramified $\mathcal{K}$-torus, and hence in this case the centralizer in $G$ of $X$ has no unramified maximal $\mathcal{K}$-torus.

Maintain the assumption that $W$ identifies with a quadratic separable field extension $\mathcal{L}$ of $\mathcal{K}$ and that $\beta$ is the norm form.

**Observation 6.3.5.** *The centralizer in* $G$ *of the torus* $SO(W, \beta)$ *is the unitary group* $M = \mathrm{Iso}(B_1 \otimes_{\mathcal{K}} \mathcal{L}, \kappa)$ *determined by the involution of second kind* $\kappa = \iota_1 \otimes \sigma$ *of the* $\mathcal{L}$*-algebra* $B_1 \otimes_{\mathcal{K}} \mathcal{L}$ *where* $\sigma$ *is the non-trivial element of* $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$*.*

With the notations of the preceding observation, $X \in \mathrm{Lie}(M)$ is geometrically distinguished (for the action of $M$). The reductive subgroup $M$ becomes the Levi factor of a parabolic subgroup of $G_{\mathcal{L}}$, though $M$ is not the Levi factor of any $\mathcal{K}$-parabolic subgroup of $G$.

If $\mathcal{L}$ is a ramified extension of $\mathcal{K}$, the reductive $\mathcal{K}$-groups $M$ has no reductive model over $\mathcal{A}$. Thus, when $\mathcal{L}$ is a ramified extension of $\mathcal{K}$, $M$ does not satisfy the conclusion of Theorem 1.3.1 for the nilpotent element $X$.

**Observation 6.3.6.** *There is a* $\mathcal{K}$*-homomorphism* $\psi : \mu_2 \to H = O(W, \beta) = \mathrm{Iso}(C, j)$ *for which* $\psi(-1)$ *acts as the non-trivial* $\mathcal{K}$*-automorphism on* $\mathcal{L} = W$ *for the natural action of* $H$ *on* $\mathcal{L} = W$*. Abusing notation, we also write* $\psi$ *for the homomorphism* $\mu_2 \xrightarrow{(1,\psi)} G_1 \times H \to G = \mathrm{Iso}(B, \iota)$ *determined by the natural mapping* $G_1 \times H \to G$*. For the natural action of* $G$ *on* $B$*, the fixed points of* $\psi(-1)$*on* $B$ *coincide with* $B_1 \otimes_{\mathcal{K}} \mathcal{L}$*. In particular, the centralizer* $C_G(\psi)$ *in* $G$ *of the image of* $\psi$ *is isomorphic to* $\mathrm{Iso}(B_1, \iota_1) \times \mathrm{Iso}(B_1, \iota_1)$*.*

It is now easy to see that $X \in \mathrm{Lie}(C_G(\psi))$ is geometrically distinguished for the action of $C_G(\psi)$; since $C_G(\psi)$ is a *split* reductive $\mathcal{K}$-group (recall that $\iota_1$ is the involution determined by a hyperbolic symmetric bilinear form), one knows that $C_G(\psi)$ is an unramified reductive group over $\mathcal{K}$. The group $C_G(\psi)$ satisfies the conclusion of Theorem 1.3.1 for the nilpotent element $X$.

## APPENDIX A. NILPOTENT ELEMENTS AND SUBGROUPS OF TYPE $C(\mu)$

Let G be a connected and reductive group over the field $\mathcal{F}$, and let $M \subset G$ be a subgroup of G of type $C(\mu)$ – see section 2.2 for an explanation of the terminology. By definition, $M = C_G^o(\psi)$ where $\psi : \mu_n \to G$ is an $\mathcal{F}$-homomorphism.

In this section, we consider the cocharacters of M and of G which are associated to a nilpotent element $X \in \text{Lie}(M)$. Since $X \in \text{Lie}(M)$, the image of the $\mu$-homomorphism $\psi$ is contained in $C_G(X)$.

In this section, we are going to prove:

**Theorem A.1.** *If $\phi$ is a cocharacter of $M$ associated to $X$ for the action of $M$, then when viewed as a cocharacter of $G$, $\phi$ is associated to $X$ for the action of $G$, as well.*

If $\phi$ is a cocharacter of M, observe that the conditions "$\phi$ is associated to X for the action of M" and "$\phi$ is associated to X for the action of G" are *geometric*; we may therefore give the proof of the Theorem after extending the base field.

Thus, for the remainder of this appendix, we are going to suppose that $\mathcal{F}$ is algebraically closed.

**Proposition A.2.** *If $H$ is linear algebraic group over the field $\mathcal{F}$ for which $H^0$ is a reductive group and if $\psi : \mu_n \to H$ is a homomorphism, there is a maximal torus $T$ of $H$ normalized by the image of $\psi$.*

*Proof.* Write $n = p^a \cdot m$ with $\gcd(p, m) = 1$; thus $\mu_n \simeq \mu_{p^a} \times \mu_m$. Since $\mu_{p^a}$ is connected, the image $\psi(\mu_{p^a})$ is contained in $H^0$. It was proved in [McN20, Theorem 3.4.1] that the image $S = \psi(\mu_{p^a})$ lies in some maximal torus T of $H^0$. Note that $C_H^0(S)$ is reductive and contains a maximal torus of H; thus it suffices to complete the proof after replacing H by $C_H(S)$. Since now the image S is contained in every maximal torus of H, it is enough to argue that the image $S_1 = \psi(\mu_m)$ normalizes some maximal torus of H. Since $\mathcal{F}$ is algebraically closed, that assertion now follows from [Ste68, Theorem 7.5]. □

**Proposition A.3.** *Let $H$ be a linear algebraic group over $\mathcal{F}$ which has a Levi decomposition; i.e. there is a reductive $\mathcal{F}$-subgroup $M \subset H$ for which $\pi_{|M} : M \to H/R_uH$ is an isomorphism, where $\pi : H \to H/R_uH$ is the quotient mapping. If $D \subset H$ is a subgroup scheme of multiplicative type, then there is $u \in (R_uH)(\mathcal{F})$ for which $uDu^{-1} \subset M$.*

*Proof.* Write $U = R_uH$, and write $\overline{D} \subset H/U$ for the image of D. It follows from [SGA3$_{II}$, Exp. XVII Prop 4.3.1] that the restriction of the quotient mapping $\pi : H \to H/U$ determines an isomorphism $\pi_{|D} : D \xrightarrow{\sim} \overline{D}$. In particular, the group scheme $E = \pi^{-1}(\overline{D})$ is an *extension* of the group scheme $\overline{D}$ of multiplicative type by the connected and $\mathcal{F}$-split unipotent group U.

Write $\gamma : H/U \to M$ for the inverse of the isomorphism $\pi_{|M} : M \to H/U$, and let $D_1 = \gamma(\overline{D})$. Then $D_1 \subset E$ and $D_1 \subset M$. It now follows from [SGA3$_{II}$, Exp. XVII Thm 5.1.1] applied to the extension E that D and $D_1$ are conjugate by an element of $U(\mathcal{F})$, as required. □

**Proposition A.4.** *Let $X$ as above.*

*(a) The image of $\psi$ centralizes some cocharacter $\phi$ associated with $X$ in $G$.*
*(b) The image of $\psi$ normalizes some maximal torus of $C_G^o(X)$.*

*Proof.* Write $C = C_G(X)$ and note that the image of $\psi$ is contained in C. Fix a cocharacter $\phi$ associated to X, and recall Proposition 3.3.2 that if $\phi$ is a cocharacter associated with X in G, the centralizer $C_\phi$ in C of the image of $\phi$ is a Levi factor of C.

Now, the image of $\psi$ is a diagonalizable subgroup of $C_G(X)$. According to Proposition A.3, this subgroup is conjugate by an element $u \in U(\mathcal{F})$ to a subgroup of the Levi factor $C_\phi$ of C. More precisely, the image of $\text{Ad}(u) \circ \psi$ is centralized by the image of $\phi$. But then, the image of $\psi$ is centralized by the image of $\text{Ad}(u^{-1}) \circ \phi$; equivalently, $\text{Ad}(u^{-1}) \circ \phi$ takes values in $M = C_G^0(\psi)$. Now, by Proposition 3.3.2(e) $\text{Ad}(u^{-1}) \circ \phi$ is a cocharacter associated with X; assertion (a) has now been proved.

Since $C_\phi$ is a Levi factor, it contains a maximal torus of C. Now (b) follows from Proposition A.2. □

Recall that for any $n \in \mathbf{Z}$, a representation $\rho : \mu_n \to GL(V)$ amounts to the data of a $\mathbf{Z}/n\mathbf{Z}$-grading of V. Indeed, recall that $\mathcal{F}[\mu_n] = \mathcal{F}[T]/\langle T^n - 1\rangle$. Now the representation $\rho$ amounts to a *comodule map* $\rho^* : V \to \mathcal{F}[\mu_n] \otimes_{\mathcal{F}} V$; for $v \in V$, write

$$\rho^*(v) = \sum_{i+n\mathbf{Z} \in \mathbf{Z}/n\mathbf{Z}} v_i \otimes T^i.$$

We obtain a $\mathbf{Z}/n\mathbf{Z}$ grading $V = \bigoplus_{i+n\mathbf{Z}\in\mathbf{Z}/n\mathbf{Z}} V_i$ by setting $V_j = \{v \in V \mid v = v_j\}$ for $j + n\mathbf{Z} \in \mathbf{Z}/n\mathbf{Z}$.

We require the following technical result, which is a slight generalization of [MS03, Lemma 24]. For the completeness and for the convenience of the reader, we repeat the proof.

**Proposition A.5.** *Let $H$ be a linear algebraic group over $\mathcal{F}$. Assume that $H^0$ is reductive, and let $\psi : \mu_n \to H$ be an $\mathcal{F}$-homomorphism. Assume that $S \subset H$ is a central torus in $H$ which is normalized by the image of $\psi$ and that $C^0_S(\psi)$ is a* maximal *central torus of $C^0_H(\psi)$. Then*

$$(C^0_H(\psi), C^0_H(\psi)) = C^0_{(H,H)}(\psi).$$

*In particular, the identity component of the centralizer in the derived group* $\operatorname{der}(H) = (H, H)$ *of the image of $\psi$ is* semisimple.

*Proof.* Write $N = C^0_{(H,H)}(\psi)$. It is clear that $(C^0_H(\psi), C^0_H(\psi)) \subset N$, and it remains to argue the reverse inclusion. For that, it is enough to argue that $N$ is semisimple. Indeed, we then have $N = (N, N)$, and since $N \subset C^0_H(\psi)$, we may deduce the required inclusion $N \subset (C^0_H(\psi), C^0_H(\psi))$.

Choose a maximal torus $T \subset H$ normalized by the image of $\psi$. The adjoint action of $\psi$ yields $\mathbf{Z}/n\mathbf{Z}$-gradings

$$\operatorname{Lie}(H) = \bigoplus_{i\in\mathbf{Z}/n\mathbf{Z}} \operatorname{Lie}(H)(i), \quad \operatorname{Lie}((H,H)) = \bigoplus_{i\in\mathbf{Z}/n\mathbf{Z}} \operatorname{Lie}((H,H))(i) \quad \text{and} \quad \operatorname{Lie}(T) = \bigoplus_{i\in\mathbf{Z}/n\mathbf{Z}} \operatorname{Lie}(T)(i),$$

and we have $\operatorname{Lie}(C^0_T(\psi)) = \operatorname{Lie}(T)(0)$, $\operatorname{Lie}(C^0_H(\psi)) = \operatorname{Lie}(H)(0)$ and $\operatorname{Lie}(C^0_{(H,H)}(\psi)) = \operatorname{Lie}((H,H))(0)$.

According to [Spr98, Cor. 8.1.6], the product mapping

$$\mu : T \times (H, H) \to H$$

is surjective. Moreover, for $(X, Y) \in \operatorname{Lie}(T) \times \operatorname{Lie}((H, H))$, [Spr98, (4.4.12)] shows that $d\mu_{(1,1)}(X, Y) = X + Y$. Now, it follows from [Cor. 7.6.4 Spr98] that $T = C_H(T)$ and thus $\operatorname{Lie}(T) = \operatorname{Lie}(H)^T$. Moreover, since $\operatorname{Lie}((H, H))$ contains each non-zero $T$-weight space of $\operatorname{Lie}(H)$, $\operatorname{Lie}(H)$ is the sum of $\operatorname{Lie}(T)$ and $\operatorname{Lie}((H, H))$ – i.e. $d\mu_{(1,1)}$ is surjective. Since this product map respects the action of the image of $\psi$, we find that $d\mu_{(1,1)} : \operatorname{Lie}(T)(i) \oplus \operatorname{Lie}((H, H))(i) \to \operatorname{Lie}(H)(i)$ is surjective for each $i \in \mathbf{Z}/n\mathbf{Z}$. In particular,

$$d\mu_{(1,1)} : \operatorname{Lie}(T)(0) \oplus \operatorname{Lie}((H, H))(0) \to \operatorname{Lie}(H)(0)$$

is surjective. This surjectivity implies that $\mu$ restricts to a dominant morphism

$$\widetilde{\mu} : C^0_T(\psi) \times N \to C^0_H(\psi).$$

Since $C^0_T(\psi)$ normalizes $N$, the image is a subgroup. Since $C^0_H(\psi)$ is connected, $\widetilde{\mu}$ is surjective; thus $C^0_H(\psi) = C^0_T(\psi)N$.

The group $N$ is reductive; let $R$ denote its maximal central torus. Now, $R$ is contained in each maximal torus of $N$; in particular $R$ is contained in $C_{T_1}(\psi)$ for some maximal torus $T_1$ of $H$ normalized by the image of $\psi$. Choosing $T = T_1$ in the preceding discussion, we find that $C^0_H(\psi) = C^0_{T_1}(\psi) \cdot N$. Thus we find that $R$ is moreover central in $C^0_H(\psi)$. But we have assumed that $C^0_S(\psi)$ to be the maximal central torus of $C^0_H(\psi)$, so we find that $R \subset C^0_S(\psi) \cap N$.

Finally, $C^0_S(\psi)$ is contained in the center $Z$ of $H$. Since $Z \cap (H, H)$ is finite – see [Spr98, (8.1.6)] – it follows that $C^0_S(\psi) \cap (H, H)$ is finite, hence also $C^0_S(\psi) \cap N$ is finite, as well. This proves that $R = 1$ so indeed $N$ is semisimple, as required. □

*Proof of Theorem A.1.* In view of the conjugacy of associated cocharacters Proposition 3.3.2, the Theorem will follow if we argue that there is a cocharacter of $M$ that is associated to $X$ both for the action of $M$ and for the action of $G$.

As was already observed, if $\phi$ is a cocharacter of $M$, the condition that $\phi$ is associated to $X$ in either $G$ or $M$ is unaffected by extension of scalars. Thus, to prove the Theorem, we may and will suppose that $\mathcal{F}$ is algebraically closed.

When $M = L$ is a Levi factor of a parabolic of $G$, this conclusion is immediate from definitions, since we can find a reductive subgroup $L_1$ for which $X \in \operatorname{Lie}(L_1)$ is distinguished, and for which $L_1$ is a Levi factor of a parabolic of $G$ and $L_1$ is a Levi factor of a parabolic of $L$.

Recall that $M = C_G^o(\psi)$ for a homomorphism $\psi : \mu_n \to G$ for some $n \geqslant 2$. Fix a maximal torus $S_0$ of $C_M(X)$. If we now set $G_1 = C_G(S_0)$ and $M_1 = C_M(S_0)$, then $G_1$ is a Levi factor of a parabolic of G, $M_1$ is a Levi factor of a parabolic of M, $M_1 = C^0_{G_1}(\psi)$ is a subgroup of $G_1$ of type $C(\mu)$, and X is distinguished in $\mathrm{Lie}(M_1)$.

Since the conclusion of the Theorem is valid for Levi factors of parabolic subgroups, a cocharacter of $M_1$ associated to X in $G_1$ is associated to X in G, and a similar statement holds for $M_1$ and M. Thus in giving the proof, we may and shall replace G by $G_1$ and M by $M_1$ and so we suppose that X is distinguished in Lie(M).

According to Proposition A.4, we may choose a cocharacter $\phi$ associated to X which is centralized by the image of $\psi$. In particular, $\phi$ is a cocharacter of M. We are going to argue that $\phi$ is associated to X in M. In view of the conjugacy of associated cocharacters Proposition 3.3.2, this will complete the proof of the Theorem. Since X is distinguished in Lie(M), and since evidently $X \in \mathrm{Lie}(M)(\phi; 2)$, in order to argue that $\phi$ is associated to X, we only must argue that the image of $\phi$ lies in the derived group M.

For this, use Proposition A.4 to choose a maximal torus S of $C_G(X)$ which is normalized by the image of $\psi$. Now, X is distinguished in the Lie algebra of $H = C_G(S)$, so by definition the image of $\phi$ is contained in $(H, H)$. Thus we see that the image of $\phi$ is contained in $C^0_{(H,H)}(\psi)$. Since X is distinguished in Lie(M), it follows that $C_S(\psi)$ is central in M. On the other hand, the connected center of M is a torus in $C_G(X)$ normalized by the image of $\psi$, we see that $C_S(\psi)$ coincides with the connected center of M.

Now Proposition A.5 implies that $(C^0_H(\psi), C^0_H(\psi)) = C^0_{(H,H)}(\psi)$, so the image of $\phi$ lies in

$$(C^0_H(\psi), C^0_H(\psi)) \subset (M, M)$$

as required. □

A result similar to Theorem A.1 was obtained in [MS03, Prop. 23] for "pseudo-Levi subgroups" M of G, though the result was only stated in *loc. cit.* for distinguished X. In general, the class of subgroups of type $C(\mu)$ is strictly larger than the class of pseudo-Levi subgroups – see the discussion in the introduction to [McN20]. The proof we have given is basically that given in [MS03], except that we have used here the result Proposition A.3 for diagonalizable group schemes deduced from [$\mathrm{SGA3}_{II}$, Exp. XVII Thm 5.1.1] rather than the result [Jan04, (11.24)], which is formulated for smooth linearly reductive groups.

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Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155, USA
*Email address*: george.mcninch@tufts.edu