FILTRATIONS AND POSITIVE CHARACTERISTIC HOWE DUALITY

GEORGE J. MCNINCH

1. Introduction

Consider a connected reductive algebraic group G defined and split over the finite field $\mathbb{F}_p$ for a prime number p. In this paper, the Jantzen filtrations of the Weyl modules for G are compared to the recently introduced filtrations of Henning Andersen associated with tilting modules

More precisely, let λ be a dominant weight. Associated with λ are a number of G modules: these include the simple module $L(\lambda)$, the Weyl module $\Delta(\lambda)$, and the indecomposable tilting module $T(\lambda)$. A G module T is a called a tilting module if both T and T^* (the contragredient module) have a filtration with factors of the form $\Delta(\mu)$ for various μ. It is known that any tilting module T may be written uniquely as a direct sum of modules $T(\mu)$ for various μ.

In order to study the tilting module T, Andersen [And97a, And97b] has introduced a space $F_\lambda(T)$ with a filtration. His construction is analogous to that for the Jantzen filtration of the Weyl module $\Delta(\lambda)$. One works with the corresponding group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, the p-adic integers. The tilting modules T and $T(\lambda)$, and the Weyl module $\Delta(\lambda)$, all arise via reduction modulo p from certain $\mathbb{Z}_p$ free $G_{\mathbb{Z}_p}$ modules; hence $F_\lambda(T)$ arises from a $\mathbb{Z}_p$ lattice. One defines a filtration of this lattice which then induces a filtration modulo p.

An initial reason for interest in these filtrations is the following: the top filtration factor of the Jantzen filtration of the Weyl module $\Delta(\lambda)$ is the simple G module $L(\lambda)$, and the dimension of the top filtration factor of the Andersen filtration of $F_\lambda(T)$ is equal to the multiplicity of $T(\lambda)$ in T.

In this paper, we generalize the situation a bit. Let $A_{\mathbb{Z}_p}$ be a $\mathbb{Z}_p$ algebra which is free of finite rank as a $\mathbb{Z}_p$ module, whose reduction modulo p, say A, is a *quasi-hereditary* algebra. (Actually, $\mathbb{Z}_p$ will be replaced by an arbitrary complete discrete valuation ring). Then A has modules analogous to the modules $L(\lambda)$, $\Delta(\lambda)$, $T(\lambda)$ and T discussed above; these modules again arise via reduction modulo p from $\mathbb{Z}_p$ free $A_{\mathbb{Z}_p}$ modules. We construct in this generality the Jantzen filtration of the Weyl module $\Delta(\lambda)$, the space $F_\lambda(T)$ introduced by Andersen, as well as the Andersen filtration of $F_\lambda(T)$. Consider the $\mathbb{Z}_p$ algebra $B_{\mathbb{Z}_p} = \mathrm{End}_{A_{\mathbb{Z}_p}}(T_{\mathbb{Z}_p})$; for certain tilting modules T the reduction modulo p, say B, is itself quasi-hereditary. The space $F_\lambda(T)$ constructed for the A module T is a B module. Provided one has a suitable notion of dual module for A, we show in Theorem 2 that the B module $F_\lambda(T)$ coincides with a Weyl module $\Delta_B(\lambda')$ for B, and that Andersen's filtration of $F_\lambda(T)$ is precisely the Jantzen filtration of $\Delta_B(\lambda')$.

The Theorem above is useful provided one has an *a priori* description of each of the algebras A and B; that is the case in the setting of reductive groups when one considers the notion of "Howe dual pairs". These pairs have been described by Donkin [Don93] and Adamovich and Rybnikov [AR96]. Roughly speaking, such a pair consists of reductive groups G,H and a G,H

Date: September 1, 2009.

bimodule T which is a tilting module for each group. Each Howe dual pair is associated to a pair of quasi hereditary $\mathbb{F}_p$ algebras $\mathcal{S}_G$, $\mathcal{S}_H$ (the generalized Schur algebras), and in fact by work of Donkin [Don86], these algebras arise via reduction modulo p from finite rank free $\mathbb{Z}_p$ algebras. The representation theory of these quasi hereditary algebras encodes part of the representation theory of G and H.

For a dual pair G and H, the pair of quasihereditary algebras $\mathcal{S}_G$ and $\mathcal{S}_H$ can be viewed as the algebras A and B associated with the tilting A module T as described above. Thus, Theorem 2 implies that the Andersen filtration $F_\lambda(T)$ computed for the G module T coincides with the Jantzen filtration of a certain H Weyl module $\Delta_H(\lambda')$.

Having made this general observation, we discuss some applications when the group H has semisimple rank 1. We point out how the constructions in [AR96] may be applied to give a simple proof of a result of Premet and Suprunenko [PS83] describing the Weyl modules for the fundamental dominant weights for the Symplectic group. Furthermore, using the coincidence of the Andersen filtration and the Jantzen filtration, we obtain straightforward parameterizations of the indecomposable summands of the following modules:

- $\bigwedge^i V \otimes \bigwedge^j V$ for $\mathrm{GL}(V)$ (see Proposition 6.3.3),
- $\bigwedge^i V$ for $\mathrm{Sp}(V)$ (see Proposition 6.3.5),
- $\bigwedge^i V \otimes \mathbb{S}$ for $\mathrm{Spin}(V)$ where $\mathbb{S}$ denotes the spin module, and V is an odd dimensional orthogonal space (see Proposition 6.3.7).

Finally, we apply the coincidence of the Jantzen and Andersen filtration to evaluate the "Jantzen sum formulae" for certain Weyl modules; we obtain in a straightforward way certain formulae, (6.4.b), first obtained by Jantzen in his thesis. This leads us to pose some questions concerning the sum formula associated with Andersen's tilting filtration.

In this paper, all rings have 1. Homomorphisms between left modules act from the right.

I would like to thank Henning Andersen, Jon Brundan, and the referee for some helpful suggestions.

2. Quasi-hereditary algebras and duality

2.1. **Basic properties of Quasi-hereditary algebras.** In what follows, k is a field. Let A be a k algebra, finite dimensional over k. Index the simple left A modules by a (necessarily finite) set Ω. For each $\lambda \in \Omega$ let $L(\lambda), P(\lambda), Q(\lambda)$ denote respectively the simple module corresponding to λ, its projective cover, and its injective hull.

Assume that $\leq$ is a given partial ordering on Ω. For $\lambda \in \Omega$, one defines the standard module, or Weyl module, $\Delta(\lambda)$ as the quotient of $P(\lambda)$ by the sum of all images $P(\mu) \to P(\lambda)$ for all $\mu \not\leq \lambda$ in Ω, and dually one defines the co-standard module $\nabla(\lambda)$ as the intersection of all kernels of maps $Q(\lambda) \to Q(\mu)$ for all $\mu \not\leq \lambda$ in Ω.

Two partial orderings $\leq$ and $\leq'$ on Ω are *equivalent* provided that the set of modules $\{\Delta(\lambda) : \lambda \in \Omega\}$ is the same when computed with respect to the two orderings, and that the analogous condition for $\{\nabla(\lambda) : \lambda \in \Omega\}$ holds.

The category of finitely generated A modules will be denoted A-mod. A module M in A-mod is said to have a Δ-filtration, or to be Δ-good, provided that there is a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with $M_i/M_{i-1} \simeq \Delta(\lambda_i)$ for some $\lambda_i \in \Omega$ and for each $i = 1, 2, \ldots, r$. The notion of ∇-filtration is analogous.

The data $(A,\Omega,\le)$ is said to be *quasi-hereditary* (for short, QH) provided that (1) $\le$ is equivalent to a total ordering on Ω, (2) the regular module ${}_AA$ has a Δ-filtration and (3) for each $\lambda\in\Omega$, $L(\lambda)$ appears as a composition factor in $\Delta(\lambda)$ precisely once.

Remark 2.1.1. Quasi-hereditary algebras were originally introduced by Cline, Parshall and Scott; their formulation of the definition was slightly different than, but equivalent to, the above one. The formulation here follows [Rin91], though there it is assumed that the set Ω is totally ordered. See [DR92] for more discussion of QH for partially ordered Ω; note that our definition implies that the partially ordered set $(\Omega,\le)$ is *adapted* in the terminology of [DR92].

If M has a Δ filtration, denote by $[M:\Delta(\mu)]_\Delta$ the multiplicity with which $\Delta(\mu)$ appears as a filtration factor in M. Define $[M:\nabla(\mu)]_\nabla$ similarly for ∇-good M.

Let ω be the full subcategory of A-mod consisting of modules having *both* a Δ-filtration and a ∇-filtration.

2.1.2. *(Ringel* [Rin91, Thm 5, Cor 5, Prop 2]*) Let $(A,\Omega,\le)$ be a QH algebra. There is a uniquely determined basic module T so that $\omega=\textbf{add}(T)$, where $\textbf{add}(T)$ is the full subcategory of A-mod consisting of all direct sums of direct summands of T.*

(a) *$T\simeq\bigoplus_{\lambda\in\Omega}T(\lambda)$, for indecomposable non-isomorphic modules $T(\lambda)$.*
(b) *$[T(\lambda):\Delta(\lambda)]_\Delta=1$, and if $[T(\lambda):\Delta(\mu)]_\Delta\ne0$ then $\mu\le\lambda$.*
(c) *Dually, $[T(\lambda):\nabla(\lambda)]_\nabla=1$, and if $[T(\lambda):\nabla(\mu)]_\nabla\ne0$ then $\mu\le\lambda$.*
(d) *$[T(\lambda):L(\lambda)]=1$, i.e. $L(\lambda)$ appears only once in a composition series for $T(\lambda)$.*

The module T is a tilting module, and the modules in ω are partial tilting modules, though occasionally terminology is abused and modules in ω are referred to as tilting modules. In particular, the result says that if module M has both a Δ and a ∇ filtration, then

$$M\simeq\bigoplus_{\lambda\in\Omega}T(\lambda)^{d_\lambda}\quad\text{for uniquely determined } d_\lambda\in\mathbb{Z}_{\ge0}.$$

(Unicity follows from the Krull-Schmidt-Azumaya theorem.) Let $[M:T(\mu)]_T=d_\lambda$ denote the multiplicity with which $T(\mu)$ appears as a summand of M; M is a full tilting module provided $d_\lambda>0$ for each $\lambda\in\Omega$.

Proposition 2.1.3. *A full tilting module Q for A is faithful.*

Proof. According to [Rin91, Lemma 6], any Δ-good A-module X embeds in a tilting module. If $a\in A$ annihilates a full tilting module, then a annihilates each $T(\lambda)$, and hence annihilates X. Since the regular representation is Δ-good, one deduces that $aA=0$, hence $a=0$. □

The fact that A is quasihereditary is reflected in a number of "homological" properties. According to [Rin91, Cor. 4], one has

$$\mathrm{Ext}^i_A(\Delta(\lambda),\nabla(\mu))=0 \text{ if } i\ne0 \text{ or } \lambda\ne\mu. \tag{2.1.a}$$

From (2.1.a) one easily obtains the following:

$$\mathrm{Ext}^i_A(T(\lambda),T(\mu))=0 \text{ if } i\ge1. \tag{2.1.b}$$

Finally, one has

$$\mathrm{Ext}^i_A(\Delta(\lambda),\Delta(\mu))=\mathrm{Ext}^i_A(\nabla(\mu),\nabla(\lambda))=0 \text{ for all } i\ge1 \text{ if } \lambda\not\le\mu; \tag{2.1.c}$$

the Ext vanishing assertions of (2.1.c) are immediate from the definitions when $i = 1$, and follow for $i > 1$ from [Rin91, Prop. 2] together with a long exact sequence argument.

For each λ, we have a non-zero homomorphism

$$c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$$

determined by the projection $\Delta(\lambda) \to \mathrm{head}\Delta(\lambda) = L(\lambda)$ and the inclusion $L(\lambda) = \mathrm{soc}\,\nabla(\lambda) \subset \nabla(\lambda)$. The following is straightforward.

Proposition 2.1.4. *Let D^λ be the k division algebra $\mathrm{End}_A(L(\lambda))$; then $\mathrm{End}_A(\Delta(\lambda)) \simeq D^\lambda$ and $\mathrm{End}_A(\nabla(\lambda)) \simeq D^\lambda$. Furthermore, the homomorphism space $\mathrm{Hom}_A(\Delta(\lambda), \nabla(\lambda))$ is a $D^\lambda - D^\lambda$ bimodule and is a 1 dimensional D^λ vector space with basis c_λ under either action.*

Proposition 2.1.5. *Let M be a partial tilting module. Then $[M : T(\lambda)]_T \neq 0$ if and only if c_λ factors through M.*

Proof. The argument given in [And98, Prop 2.5] works in this generality. The only point to modify is to replace a λ weight vector in $T(\lambda)$, as in the argument in [And98], with a lift of a vector generating the unique section of $T(\lambda)$ isomorphic to $L(\lambda)$ (that $T(\lambda)$ has precisely one such composition factor follows from 2.1.2). □

Let T be a full tilting A module, and form the algebra $A' = \mathrm{End}_A(T)$. The functor

$$F : A\text{-mod} \to A'\text{-mod} \quad \text{via } N \mapsto \mathrm{Hom}_A(T, N) \tag{2.1.d}$$

was studied by Ringel in [Rin91]; he proves

2.1.6. ([Rin91, Thm 6]) *Let $(A, \Omega, \leq)$ be QH, and let T be a full tilting module. Let $(\Omega', \leq)$ be the opposite partially ordered set, with the defining order reversing bijection $\Omega \to \Omega'$ denoted $\lambda \mapsto \lambda'$. The set Ω' indexes the simple $A' = \mathrm{End}_A(T)$ modules, and $(A', \Omega', \leq)$ is QH. Furthermore, using some obvious notational conventions, one has*

$$P'(\lambda') = FT(\lambda), \quad L'(\lambda') = \text{head } P'(\lambda'), \quad \text{and } \Delta'(\lambda') = F\nabla(\lambda)$$

for each $\lambda \in \Omega$. The functor F defines an equivalence of categories between $\mathcal{F}(\nabla)$, the full subcategory of A-mod consisting of ∇-good modules, and $\mathcal{F}(\Delta')$, the Δ'-good A' modules.

Remark 2.1.7. The algebra A' obtained above from the full tilting module T is in general only Morita equivalent to the one constructed by Ringel in [Rin91, Thm 6], as there the basic tilting module T is used. The algebra A' is sometimes referred to as the "Ringel dual" of A.

Remark 2.1.8. The proof of [Rin91, Thm 6] makes it clear that the radical (the unique maximal A' submodule) of $\Delta'(\lambda')$ may be identified with the k vector subspace of $\mathrm{Hom}_A(T, \nabla(\lambda))$ spanned by all ϕ which factor through some $T(\mu)$ with $\mu > \lambda$; i.e. for which there is a commuting diagram of A modules

$$\begin{array}{ccc} T & \longrightarrow & T(\mu) \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \nabla(\lambda) \end{array} .$$

Let $y : T \to \nabla(\lambda)$. If $T(\lambda) \xrightarrow{s} T$ is a split injection, and the restriction of y to the image of s is non-0, then y generates $\Delta'(\lambda')$ as an A' module. Indeed, it suffices to observe that y is not in the radical. Suppose otherwise, and assume y is a linear combination of maps which factor through $T(\mu_i)$ where $\mu_i > \lambda$ for $1 \leq i \leq k$. Then one would have a non-0 homomorphism

$$\Delta(\lambda) \subseteq T(\lambda) \to \oplus_i T(\mu_i) \to \nabla(\lambda)$$

contrary to Proposition 2.1.5.

Consider the bilinear pairing $\varepsilon : \mathrm{Hom}_A(\Delta(\lambda), T) \times \mathrm{Hom}_A(T, \nabla(\lambda)) \to \mathrm{Hom}_A(\Delta(\lambda), \nabla(\lambda)) = D^\lambda c_\lambda$ given by composition.

Proposition 2.1.9. *Let y be a generator for $\mathrm{Hom}_A(T, \nabla(\lambda)) = \Delta'(\lambda')$ as A' module as in Remark 2.1.8.*

(a) $\varepsilon(z, y) = 0$ *for each $z : \Delta(\lambda) \to T$ which factors as $z = z_1 \circ z_2 : \Delta(\lambda) \xrightarrow{z_1} T(\mu) \xrightarrow{z_2} T$ for some $\mu > \lambda$.*

(b) $\varepsilon(y', y) = c_\lambda$ *for some $y' : \Delta(\lambda) \to T$. In particular,* $\varepsilon(-, y) : \mathrm{Hom}_A(\Delta(\lambda), T) \to D^\lambda c_\lambda$ *is surjective.*

Proof. For part (a), $\varepsilon(z, y)$ determines an A homomorphism $\Delta(\lambda) \to T(\mu) \to T \to \nabla(\lambda)$; this must vanish by Proposition 2.1.5. As to (b), the same Proposition shows that c_λ must factor through $T(\lambda)$ hence there are maps y, y' with $\varepsilon(y, y') = c_\lambda$; the result follows since $T(\lambda)$ is an A summand of T. Evidently $\varepsilon(d \circ y', y) = dc_\lambda$ for $d \in D^\lambda$; this yields the surjectivity. □

Finally, we record the following useful formula.

Proposition 2.1.10. *(*[Don93, Lemma 3.1]*) For $\lambda, \mu \in \Omega$,* $[T(\lambda) : \nabla(\mu)]_\nabla = [\nabla'(\mu') : L'(\lambda')]$.

2.2. **Algebras with duality.** Let A and B be rings. If $\phi : A \to B$ is a ring homomorphism, one gets a functor $\mathcal{F}_\phi$ from B-mod to A-mod; for a B module M the underlying Abelian group of $\mathcal{F}_\phi(M)$ is M, with A acting through ϕ. On morphisms, this functor is the identity. Of course, if ϕ is an isomorphism, $\mathcal{F}_\phi$ is an equivalence of categories.

Let A be a finite dimensional k algebra. For a finitely generated left A module M, the ordinary linear dual $M^* = \mathrm{Hom}_k(M, k)$ is a right A module, or, what is the same, a left A^{op} module; namely, given $a^{\mathrm{op}} \in A^{\mathrm{op}}$ corresponding to $a \in A$, and $\phi \in M^*$, then $(x)(a^{\mathrm{op}} * \phi) = (ax)\phi$ for $x \in M$. Given a homomorphism of A modules $\phi : M \to N$, the dual map $\phi^* : N^* \to M^*$ is A^{op}-linear. In this way one gets a contravariant equivalence of categories $\mathcal{D} : A\text{-mod} \to \text{mod-}A = A^{\mathrm{op}}\text{-mod}$ via $\mathcal{D}(M) = M^*$.

Now suppose that $\iota : A \to A$ is an antiautomorphism. We will use the antiautomorphism ι to define a "duality" $(-)^\vee$ on the category A-mod. Regarding ι as an isomorphism $\iota : A \to A^{\mathrm{op}}$, one gets the covariant equivalence $\mathcal{F}_\iota : A^{\mathrm{op}}\text{-mod} \to A\text{-mod}$. Let $\mathcal{E}_\iota = \mathcal{F}_\iota \circ \mathcal{D}$ be the contravariant equivalence A-mod $\to$ A-mod obtained by composition. We abbreviate $\mathcal{E}_\iota(M) = M^\vee$ when convenient. If $\phi : M \to N$ is A linear, then $\phi^\vee : N^\vee \to M^\vee$ coincides with the vector space dual map ϕ^*.

Remark 2.2.1. If $\iota^2 = 1$, then $(M^\vee)^\vee \simeq M$; in general, though, $(M^\vee)^\vee \simeq \mathcal{F}_{\iota^2}(M)$. Despite this, we persist in referring to $M^\vee$ as a "dual" module.

Let $\beta(-,-) : M \times N \to k$ be a bilinear pairing. We say that β is *A-equivariant* if $\beta(x, ay) = \beta(\iota(a)x, y)$ for every $a \in A$, $x \in M$, $y \in N$.

Lemma 2.2.2. *Let A be an algebra with antiautomorphism ι. Let M and N be A modules, and suppose that $\beta : M \times N \to k$ is a non-0 equivariant pairing.*

(a) *Let $M' \subset M$ be an A submodule. If β is non-degenerate, the canonical vector space isomorphism $(M')^{\perp} \xrightarrow{\simeq} (M/M')^{\vee}$ is A-linear.*
(b) *Assume that M and N have respective unique maximal submodules M_{max}, N_{max}, and suppose that there is an element $y \in N \setminus N_{max}$ such that $y \in M_{max}^{\perp}$. Then the simple modules $(M/M_{max})^{\vee}$ and N/N_{max} are isomorphic, $M_{max} = N^{\perp}$, and $N_{max} = M^{\perp}$.*

Proof. (a) is immediate from the definitions. For (b), note first that any non-zero quotient of M or N has also a unique maximal submodule. Since the form is non-zero, we may replace M with $M/N^{\perp}$ and N with $N/M^{\perp}$ and assume that β is non-degenerate; the image of y in $N/M^{\perp}$ continues to enjoy the appropriate hypothesis. By equivariance, $M_{\mathrm{max}}^{\perp}$ is an A submodule of N containing y, hence $M_{\mathrm{max}}^{\perp} = N$. By non-degeneracy, this implies $M_{\mathrm{max}} = 0$ hence M is simple. But then by (a), we get $N \simeq M^{\vee}$ is simple as well. ☐

For an A-module M, consider the k algebra $A'_M = \mathrm{End}_A(M)$. Suppose $\gamma : M \xrightarrow{\simeq} M^{\vee}$ is an A-module isomorphism; then γ determines an algebra isomorphism $c_\gamma : A'_{M^{\vee}} \to A'_M$ given by $c_\gamma(f) = \gamma^{-1} \circ f \circ \gamma$.

For any A module N, one obtains a vector space isomorphism $\Phi : \mathrm{Hom}_A(N,M) \to \mathrm{Hom}_A(M,N^{\vee})$ via

$$\mathrm{Hom}_A(N,M) \xrightarrow{\mathcal{E}_\iota} \mathrm{Hom}_A(M^{\vee},N^{\vee}) \xrightarrow{\gamma^*} \mathrm{Hom}_A(M,N^{\vee}). \tag{2.2.a}$$

Symbolically, $\Phi(f) = \gamma \circ f^{\vee}$. $\mathrm{Hom}_A(N,M)$ is naturally a left module for ${A'_M}^{\mathrm{op}}$ and $\mathrm{Hom}_A(M,N^{\vee})$ is naturally a left module for A'_M. We would like to regard both the left and right hand term of (2.2.a) as A'_M modules; we seek an antiautomorphism ι' of A'_M, which we regard as an isomorphism $A'_M \to {A'_M}^{\mathrm{op}}$, for which the following holds.

Lemma 2.2.3. *There is an antiautomorphism ι' of A'_M such that*

$$\mathcal{F}_{\iota'}(\mathrm{Hom}_A(N,M)) \xrightarrow{\Phi} \mathrm{Hom}_A(M,N^{\vee})$$

is an isomorphism of A'_M modules.

Proof. Since $\mathcal{E}_\iota$ is a contravariant equivalence, it induces an anti-isomorphism $\mathcal{E}_\iota : \mathrm{End}_A(M) \to \mathrm{End}_A(M^{\vee})$; in fact, this anti-isomorphism is given by $f \mapsto f^{\vee} = f^*$. The composition $c_\gamma \circ \mathcal{E}_\iota$ is an antiautomorphism $\iota^{\vee}$ of A'_M; for $a \in A'_M$, $\iota^{\vee}(f) = \gamma \circ a^{\vee} \circ \gamma^{-1}$.

We take $\iota' = (\iota^{\vee})^{-1}$ and check that it has the required property. Well, evidently it is a linear isomorphism. Suppose $a \in A'_M$ and $\psi \in \mathcal{F}_{\iota'}(\mathrm{Hom}_A(N,M))$. The action of a is via $a * \psi = \psi \circ \iota'(a)$. So $\Phi(a * \psi) = \gamma \circ (\psi \circ \iota'(a))^{\vee} = \gamma \circ (\iota'(a))^{\vee} \circ \psi^{\vee}$. On the other hand, $a * \Phi(\psi) = a \circ \gamma \circ \psi^{\vee}$. Thus, we need to see that $\gamma \circ (\iota'(a))^{\vee} = a \circ \gamma$, or that $(\iota'(a))^{\vee} = \gamma^{-1} \circ a \circ \gamma$, which is clear. ☐

2.3. **Quasi-hereditary algebras with duality.** Suppose now that $(A, \Omega, \leq)$ is QH. For each $\lambda \in \Omega$, evidently $L(\lambda)^{\vee}$ is simple; it follows that there is a function $w : \Omega \to \Omega$ with the property that $L(\lambda)^{\vee} \simeq L(w\lambda)$. It is straightforward to see that w must be *bijective* (the inverse antiautomorphism $\iota^{-1} : A \to A$ induces w^{-1}).

We say that the antiautomorphism ι is *compatible with the order* $\leq$ provided that w is an automorphism of the partially ordered set $(\Omega, \leq)$, i.e. that $\mu \leq \lambda$ if and only if $w\mu \leq w\lambda$.

We say that the data $(A, \Omega, \leq, \iota)$ comprise a *QH algebra with duality* in case $(A, \Omega, \leq)$ is a QH algebra and ι is compatible with $\leq$ in the above sense.

Remark 2.3.1. In order to see that w is an automorphism of Ω, it suffices to check that $w\mu \leq w\lambda$ whenever $\mu \leq \lambda$. Indeed, for any $n \geq 0$ one has $w^n\mu \leq w^n\lambda$. Since Ω is a finite set, w has finite order, so the same is true of w^{-1}.

Lemma 2.3.2. *Let $(A,\Omega,\leq,\iota)$ be a QH algebra with duality. With notation as above, one has for each $\lambda \in \Omega$*

(a) $P(\lambda)^\vee \simeq Q(w\lambda)$ *and* $Q(\lambda)^\vee \simeq P(w\lambda)$
(b) $\Delta(\lambda)^\vee \simeq \nabla(w\lambda)$ *and* $\nabla(\lambda)^\vee \simeq \Delta(w\lambda)$
(c) $T(\lambda)^\vee \simeq T(w\lambda)$.

Proof. It is easy to check that $P(\lambda)^\vee$ is injective, indecomposable, and has socle $L(w\lambda) = L(\lambda)^\vee$. Similarly, $Q(\lambda)^\vee$ is projective, indecomposable and has head $L(w\lambda)$; (a) follows immediately.

As to (b), if $\tau \in \Omega$, let $\Omega(\not\leq \tau) = \{\mu \in \Omega : \mu \not\leq \tau\}$. The above discussion shows that $w\Omega(\not\leq \tau) = \Omega(\not\leq w\tau)$. For $\mu \not\leq \tau$, let $I_\mu(\tau) \leq P(\tau)$ be the sum of all images of maps $P(\mu) \to P(\lambda)$, and let $J_\mu(\tau) \leq Q(\tau)$ be the intersection of all kernels of maps $Q(\tau) \to Q(\mu)$. Part (a) implies that there is a non-degenerate pairing $P(\lambda) \times Q(w\lambda) \to k$, and that under this pairing, $I_\mu(\lambda)^\perp = J_{w\mu}(w\lambda)$.

For this pairing, duality yields

$$\left(\sum_{\mu\in\Omega(\not\leq\lambda)} I_\mu(\lambda)\right)^\perp = \bigcap_{\mu\in\Omega(\not\leq\lambda)} J_{w\mu}(w\lambda) = \bigcap_{\tau\in\Omega(\not\leq w\lambda)} J_\tau(w\lambda) = \nabla(w\lambda);$$

Thus $\Delta(\lambda)^\vee = \left(P(\lambda)/\sum_{\mu\in\Omega(\not\leq\lambda)} I_\lambda(\mu)\right)^\vee \simeq \nabla(w\lambda)$ by Lemma 2.2.2. The remaining isomorphism of (b) follows from Remark 2.2.1 by taking the "double dual".

Finally, (c) follows from (b) together with [Rin91, Prop 2]. □

Lemma 2.3.3. *Let $(A,\Omega,\leq,\iota)$ be a QH algebra with duality. Let T be a full tilting module, and suppose that $T \simeq T^\vee$. Let ι' be the antiautomorphism of the Ringel dual $A' := A'_T$ determined as in Lemma 2.2.3. For each $\lambda \in \Omega$ there are isomorphisms $\mathcal{F}_{\iota'}(\mathrm{Hom}_A(\Delta(\lambda),T)) \simeq \mathrm{Hom}_A(T,\nabla(w\lambda)) \simeq \Delta'((w\lambda)')$ as left A' modules.*

Proof. The antiautomorphism ι' was chosen so that $\mathcal{F}_{\iota'}(\mathrm{Hom}_A(\Delta(\lambda),T)) \simeq \mathrm{Hom}_A(T,\Delta(\lambda)^\vee)$ as A' modules, so the first isomorphism follows from Lemma 2.3.2(b). The second isomorphism is obtained from Ringel's result 2.1.6. □

Theorem 1. *Suppose that $(A,\Omega,\leq,\iota)$ is a QH algebra with duality. Let T be a full tilting module for which $T \simeq T^\vee$. Then the Ringel dual $(A' = A'_T,\Omega',\leq,\iota')$ is a QH algebra with duality, where ι' is determined as in Lemma 2.2.3.*

Furthermore, if $w : \Omega \to \Omega$ and $w' : \Omega' \to \Omega'$ are the permutations induced by ι and ι', then $w'(w\lambda)' = \lambda'$ for all $\lambda \in \Omega$.

Proof. In view of Ringel's theorem 2.1.6, the entire result will be known provided we show that $L'((w\lambda)')^\vee \simeq L'((\lambda')$ for $\lambda \in \Omega$. Indeed, given this, one readily verifies that w' is compatible with the order on Ω', and that $w'(w\lambda)' = \lambda'$ for all $\lambda \in \Omega$.

Consider the evaluation pairing ε of Proposition 2.1.9 and put $\beta(-,-) = \mathrm{tr}(\varepsilon(-,-))$ where $\mathrm{tr} = \mathrm{tr}_{D/k}$ is the reduced trace as in [Rei75, §9a]. In view of Proposition 2.1.9(b), it follows from [Rei75, Theorem 9.9] that β is non-0.

Lemma 2.3.3 permits one to view β as a pairing on

$$\Delta'((w\lambda)') \times \Delta'(\lambda') \simeq \mathcal{F}_{\iota'}(\mathrm{Hom}_A(\Delta(\lambda),T)) \times \mathrm{Hom}_A(T,\nabla(\lambda));$$

it is straightforward to see that β is A' equivariant.

The isomorphism $L'((w\lambda)')^{\vee} \simeq L'(\lambda')$ now follows from Lemma 2.2.2(b) in view of Proposition 2.1.9(a) and Remark 2.1.8. □

3. R FORMS OF A QH ALGEBRA

Let R denote a complete discrete valuation ring with maximal ideal $\mathfrak{m} = R\pi$. Put $k^{(i)} = R/\mathfrak{m}^i$ for $i \geq 1$; in particular, $k = k^{(1)}$ is the residue field. The field of fractions of R will be denoted K. The π-adic valuation of K will be denoted by ν_π or $\nu_\mathfrak{m}$.

In this section, we discuss some of the consequences of the existence of an R form of the QH algebra A.

3.1. Let A_R be an admissible R algebra; i.e. one which is free of finite rank as an R module. Put $A_K = A_R \otimes_R K$, $A^{(i)} = A_R \otimes_R k^{(i)} \simeq A_R/\pi^i A_R$ and $A = A^{(1)}$.

We will say that an A_R module M_R is *admissible* in case it is a finite rank free R module. Similarly, an $A^{(i)}$ module $M^{(i)}$ is admissible if it is a finite rank free $k^{(i)}$ module. For $j \geq 1$, we say that an admissible A_R or $A^{(j+k)}$ $(k \geq 0)$ module X is a *lift* of the $A^{(j)}$ module $X/\pi^j X$.

For an A-module M, denote by $M^{(i)}$ an $A^{(i)}$ lift of M. Of course, $M^{(1)} = M$. In general, such a module $M^{(i)}$ need not exist (for $i > 1$), nor need it be unique. On the other hand, if $M^{(i)}$ exists for all $i > 0$, and these modules are *coherent* in the sense that $M^{(i)}$ is a lift of $M^{(i-1)}$ for $i \geq 2$ then $M_R = \varprojlim M^{(i)}$ is an admissible lift to A_R of M.

If M_R and N_R are A_R modules for which $M_R/\pi^k M_R \simeq N_R/\pi^k N_R$ as $A^{(k)}$ modules for all $k \geq 1$, [CR81, 30.15] shows that $M_R \simeq N_R$. This proves: If the A module M has a uniquely determined $A^{(i)}$ lift $M^{(i)}$ for each $i \geq 1$, and the $M^{(i)}$ form a coherent system of lifts, then the A_R lift M_R is uniquely determined as well.

Lemma 3.1.1. *Let M and N be A modules, let $M^{(i)}$ be an $A^{(i)}$ lift of M, and let M_R be an A_R lift of M. For each $j \geq 0$, there are isomorphisms*

(a) $\mathrm{Ext}^j_{A^{(i)}}(M^{(i)},N) \simeq \mathrm{Ext}^j_A(M,N)$, *and*

(b) $\mathrm{Ext}^j_{A_R}(M_R,N) \simeq \mathrm{Ext}^j_A(M,N)$.

Proof. If X is an $A^{(i)}$ module, notice that $A \otimes_{A^{(i)}} X \simeq k \otimes_{k^{(i)}} X$ as $k^{(i)}$ modules. If X is an admissible $A^{(i)}$ module, then a free (or projective) resolution of X as an $A^{(i)}$ module is also a free resolution of X as a $k^{(i)}$ module. Since such a sequence is $k^{(i)}$ split, one gets for an *admissible* $A^{(i)}$ module X that $\mathrm{Tor}^{A^{(i)}}_p(A,X) = X/\pi X$ if $p = 0$, and is 0 otherwise.

Considered as an A_R module, A has a projective resolution $0 \to A_R \xrightarrow{\cdot\pi} A_R \to A \to 0$. For any A_R module X, this yields $\mathrm{Tor}^{A_R}_p(A,X) = X/\pi X$ if $p = 0$, $= X(\pi)$ if $p = 1$, and is 0 otherwise, where $X(\pi)$ denotes the π torsion submodule of X. If X is admissible, hence R-free, one has $X(\pi) = 0$

For a homomorphism of rings $\phi : \Lambda \to \Gamma$ and modules ${}_\Lambda X$ and ${}_\Gamma Y$, [CE56, XVI §6 Case 3] shows that there is a first quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q_\Gamma(\mathrm{Tor}^\Lambda_p(\Gamma,X),Y) \implies \mathrm{Ext}^{p+q}_\Lambda(X,Y).$$

Take $\Gamma = A$ and $Y = N$. For part (a), take $\Lambda = A^{(i)}$ and $X = M^{(i)}$, and for (b) take $\Lambda = A_R$ and $X = M_R$. In each case the above Tor calculations show that the spectral sequence collapses, giving at once the desired isomorphisms. □

If an $A^{(i)}$ lift $M^{(i)}$ of the A module M exists, the lemma shows $\mathrm{Ext}^j_{A^{(i)}}(M^{(i)}, M) \simeq \mathrm{Ext}^j_A(M, M)$; denote this group by $e^j(M)$.

Proposition 3.1.2. *Let M be an A-module.*

(a) *Given a module $M^{(i)}$, the obstruction to the existence of a lift $M^{(i+1)}$ of $M^{(i)}$ is a cohomology class $\alpha(i,M) \in e^2(M)$. If each of these classes vanish, then there is a coherent system $\{M^{(i)}\}_{i\geq 1}$ of lifts, and hence an admissible A_R lift M_R of M.*

(b) *Assume that $\vec{\alpha}(M) = (\alpha(i,M))_{i\geq 0} = \vec{0}$ and that $M^{(i)}$ is a fixed lift of M to $A^{(i)}$. Then the isomorphism classes of lifts $M^{(i+1)}$ of $M^{(i)}$ are parameterized by $e^1(M)$. If $e^1(M) = 0$, each $M^{(i)}$ is uniquely determined, and hence M_R is uniquely determined.*

Remark 3.1.3. This result for group algebras is originally due to Green, and is proved in [Ben91, Theorem 3.7.7]; however the proof given there does not (quite) generalize. We provide the following sketch.

Sketch. Suppose inductively that one has constructed a $k^{(i)}$-free $A^{(i)}$ module $M^{(i)}$ which lifts M. Let $\cdots P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M^{(i)} \to 0$ be the first few terms of a projective (or free) resolution of $M^{(i)}$ as an $A^{(i)}$ module. The modules P^s lift to projective (or free) modules $\widehat{P^s}$ for $A^{(i+1)}$, and one can construct a commuting diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & \ker\psi & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^1/\pi P^1 & \xrightarrow{\overline{d^1}} & P^0/\pi P^0 & \xrightarrow{\overline{d^0}} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \psi & & \\
\cdots & \longrightarrow & \widehat{P^1} & \xrightarrow{\widehat{d^1}} & \widehat{P^0} & \xrightarrow{\widehat{d^0}} & \widehat{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^1 & \xrightarrow{d^1} & P^0 & \xrightarrow{d^0} & M^{(i)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns, where $\widehat{M} = \mathrm{coker}(\widehat{d^1})$. Write $\beta = \widehat{d^1}$. Note there is some choice of the map β; let β be a fixed choice. The conditions required for the map $\widehat{d^1}$ imply that $\widehat{d^1} = \beta'$ yields an exact commutative diagram if and only if $\beta' = \beta + \pi^i \eta$, where $\eta : P^1 \to P^0/\pi P^0$ is $A^{(i)}$ linear.

The candidate for $M^{(i+1)}$ is $\widehat{M} = \mathrm{coker}\,\beta$; we just need to understand when $\widehat{M}$ is $k^{(i+1)}$ free, or equivalently when the map ψ is injective. Well, by the snake lemma one has an exact sequence

$$0 \to \ker \overline{d^1} \to \ker \widehat{d^1} \to \ker d^1 \xrightarrow{\delta} M \xrightarrow{\psi} \widehat{M} \to M^{(i)} \to 0;$$

evidently ψ is injective just in case $\delta = 0$; of course, $\delta = \delta(\beta)$ depends on the choice of β. The map δ determines an element $\alpha(i,M) \in e^2(M)$; the condition $\alpha(i,M) = 0$ means that there is some $A^{(i)}$ linear map $\tilde{\eta} : P^1 \to M$ so that $\tilde{\eta}\mid_{\ker d^1} = \delta$. Since P^1 is projective, there is a lift $\eta : P^1 \to P^0/\pi P^0$ such that $\overline{d^0} \circ \eta = \tilde{\eta}$. Put $\beta' = \beta + \pi^i \eta$; one now checks that $\delta(\beta') = 0$ proving that for suitable choice of β the module $\widehat{M}$ is $k^{(i+1)}$ free.

The reader is left to verify that the class $\alpha(i,M)$ doesn't depend on the choice of β. Furthermore, if $\widehat{d^1} = \beta$ is a choice for which $\widehat{M}$ is $k^{(i+1)}$ free, then all such choices of $\widehat{d^1}$ are obtained as $\beta + \pi^i \eta$ where $\overline{d^0} \circ \eta = \tilde{\eta} : P^1 \to P_0/\pi P_0 \to M$ is a cocycle, and the reader is invited to verify that this identifies the isomorphism classes of $M^{(i+1)}$ with $e^1(M)$. □

For the remainder of the section, let the simple A modules be indexed by the set Ω, and assume given an ordering on Ω for which (A,Ω) is a quasi-hereditary algebra.

3.2. According to (2.1.b) and (2.1.c), one has $e^j(T(\lambda)) = e^j(\Delta(\lambda)) = e^j(\nabla(\lambda)) = 0$ for $j \geq 1$, so Proposition 3.1.2 yields uniquely determined admissible A_R lifts $T_R(\lambda), \Delta_R(\lambda), \nabla_R(\lambda)$. Moreover, as $e^2(Q) = 0$, there is an admissible A_R lift Q_R of any (partial) tilting A module Q. Since R is complete, the Krull-Schmidt-Azumaya theorem is valid for A_R modules [CR81, Theorem 6.12]; it implies that Q_R may be written as a direct sum of indecomposable modules $T_R(\lambda)$ with uniquely determined multiplicities.

3.3. Let M and N be A modules with $\vec{\alpha}(M) = 0$ and $\vec{\alpha}(N) = 0$. Let M_R and N_R denote admissible A_R lifts of M and N. Consider the exact sequence of A_R modules

$$0 \to N_R \xrightarrow{\cdot\pi} N_R \to N \to 0.$$

Applying the functor $\operatorname{Hom}_{A_R}(M_R, -)$ one obtains a long exact sequence; in particular, the following is exact for $p \geq 0$

$$\operatorname{Ext}^p_{A_R}(M_R,N_R) \xrightarrow{\cdot\pi} \operatorname{Ext}^p_{A_R}(M_R,N_R) \to \operatorname{Ext}^p_{A_R}(M_R,N) \to \operatorname{Ext}^{p+1}_{A_R}(M_R,N_R) \qquad (3.3.a)$$

One has:

Proposition 3.3.1. *For $p \geq 0$, $\operatorname{Ext}^p_A(M,N) = 0$ implies $\operatorname{Ext}^p_{A_R}(M_R,N_R) = 0$.*

Proof. Lemma 3.1.1 shows that $\operatorname{Ext}^p_{A_R}(M_R,N) = 0$. Exactness of the sequence (3.3.a) implies that multiplication by π on $\operatorname{Ext}^p_{A_R}(M_R,N_R)$ is surjective, and Nakayama's lemma gives the conclusion of the Proposition. □

Proposition 3.3.2. *If $\operatorname{Ext}^1_A(M,N) = 0$, then $\operatorname{Hom}_{A_R}(M_R,N_R) \otimes_R k \simeq \operatorname{Hom}_A(M,N)$.*

Proof. Proposition 3.3.1 shows that $\operatorname{Ext}^1_{A_R}(M_R,N_R) = 0$, so the result follows from (3.3.a). □

3.4. Proposition 3.3.1 together with (2.1.a) (see also [And97a, (1.2)]) show that

$$\operatorname{Ext}^i_{A_R}(\Delta_R(\lambda), \nabla_R(\mu)) = 0 \text{ if } i \neq 0 \text{ or } \lambda \neq \mu\,. \qquad (3.4.a)$$

Denote by D^λ the k division algebra $\operatorname{End}_A(L(\lambda))$. Then $\operatorname{End}_A(\Delta(\lambda)) \simeq D^\lambda$ and $\operatorname{End}_A(\nabla(\lambda)) \simeq D^\lambda$, as in Proposition 2.1.4. Put $D^{\lambda,1}_R = \operatorname{End}_{A_R}(\Delta_R(\lambda))$ and $D^{\lambda,2}_R = \operatorname{End}_{A_R}(\nabla_R(\lambda))$; by Proposition 3.3.2 and (2.1.c) one obtains

$$D^{\lambda,i}_R/\pi D^{\lambda,i}_R \simeq D^\lambda. \qquad (3.4.b)$$

That the endomorphism algebras $D_R^{\lambda,i}$ are local follows from (3.4.b); this implies that $\Delta_R(\lambda)$ and $\nabla_R(\lambda)$ are indecomposable A_R modules.

Proposition 2.1.4 shows that the D^λ-D^λ bimodule $\mathrm{Hom}_A(\Delta(\lambda),\nabla(\lambda))$ is 1 dimensional (under either action), with basis $c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$. Let $c_\lambda^R : \Delta_R(\lambda) \to \nabla_R(\lambda)$ be a fixed lift of c_λ. Proposition 3.3.2 shows that $\mathrm{Hom}_{A_R}(\Delta_R(\lambda),\nabla_R(\lambda))$ is generated by c_λ^R as both a left module for $D_R^{\lambda,2}$ a right module for $D_R^{\lambda,1}$. Furthermore, c_λ^R is not a torsion element. So the homomorphism space is even free of rank 1 under each action.

Let A_K, $\Delta_K(\lambda)$, etc be obtained by extension of scalars, and suppose that A_K is a *semisimple* K algebra. Since $\Delta_R(\lambda)$ is A_R indecomposable, $\Delta_K(\lambda)$ is a simple A_K module; similarly, $\nabla_K(\lambda)$ is a simple A_K module. Furthermore, the non-zero homomorphism $c_\lambda^K = c_\lambda^R \otimes 1_K$ is an A_K isomorphism between these simple modules, and c_λ^K induces an isomorphism $D_K^{\lambda,1} \simeq D_K^{\lambda,2}$.

Thus, $D_R^{\lambda,i}$ are R-orders in a K-division algebra. In fact, they are isomorphic R-algebras. This assertion follows from the following:

Proposition 3.4.1. *Let D be a K division ring, let $\Lambda_R \subset D$ be an R-order such that $\Lambda_R/\pi\Lambda_R$ is a skew field. Then Λ_R is the unique maximal order in D, π is a prime element of Λ_R and D is unramified over K. In particular, if k is finite, D is commutative.*

Proof. Let $\Theta_R \subset D$ be the unique maximal order in D (see [Rei75, Theorem 12.8]). One has then an R-algebra homomorphism $\Lambda_R/\pi\Lambda_R \to \Theta_R/\pi\Theta_R$. Since $\Lambda_R/\pi\Lambda_R$ is a skew field, this map must be injective; by dimension reasons it is then an isomorphism. It follows that the inclusion $\Lambda_R \subset \Theta_R$ is an isomorphism. Since $\Lambda_R/\pi\Lambda_R$ is a skew field, π is a prime element and D is unramified by [Rei75, Theorem 13.3]. The assertion when k is finite follows from [Rei75, Theorem 14.3]. □

The choice of c_λ^R determines an isomorphism of $D_R^{\lambda,1}$ and $D_R^{\lambda,2}$; denote by D_R^λ this R-algebra.

Proposition 3.4.2. *Assume that A_K is a semisimple K algebra. Let $\lambda \in \Omega$.*

(a) *The A modules $\Delta(\lambda)$ and $\nabla(\lambda)$ have the same composition factors, and the R modules $\Delta_R(\lambda)$, $\nabla_R(\lambda)$ have the same rank.*

(b) *Let Q be a partial tilting A module. Then $[Q:\Delta(\lambda)]_\Delta = [Q:\nabla(\lambda)]_\nabla$.*

Proof. For (a), the argument given in [Ser77, Theorem 32] works *mutatis mutandum.* As to (b), the notion of QH guarantees that the sets of classes $\{[\Delta(\mu)] : \mu \in \Omega\}$, and $\{[\nabla(\mu)] : \mu \in \Omega\}$ form $\mathbb{Z}$ bases for the Grothendieck group of A. (a) implies that $[\Delta(\mu)] = [\nabla(\mu)]$ for each μ and (b) follows. □

3.5. Suppose that ι_R is an antiautomorphism $A_R \to A_R$. As in section 2.2, given an A_R module M_R, we may form the dual module $M_R{}^\vee$. Observe that $\iota = \iota_R \otimes 1_k$ is an antiautomorphism of A; when M_R is admissible, it is straightforward to see that $M_R{}^\vee \otimes_R k \simeq M^\vee$ as A modules. Furthermore, if M_R is admissible, $M_R{}^\vee$ is also admissible.

Suppose that M is an A module with $e^1(M) = e^2(M) = 0$. Then M has a unique A_R lift by Proposition 3.1.2. If $M^\vee \simeq M$, then by uniqueness of the lift, one has $M_R{}^\vee \simeq M_R$.

Suppose A is QH and ι is compatible with the order on Ω, and let T be a self-dual tilting module for A. According to Proposition 3.3.2, $A'_R = \mathrm{End}_{A_R}(T_R)$ is an R form of the QH algebra $A' = \mathrm{End}_A(T)$, and the construction in Lemma 2.2.3 is easily seen to work "over R"; it yields an antiautomorphism ι'_R of A'_R such that $\iota' = \iota'_R \otimes 1$ is the antiautomorphism of Theorem 1.

4. Andersen and Jantzen filtrations

Throughout this section, we keep the notation of the previous section; we assume that A_R is an admissible R-algebra, that $(A = A_R/\pi A_R, \Omega, \leq)$ is a QH k algebra for a suitable partial ordering on Ω, and that A_K is a semisimple K algebra.

4.1. The Jantzen filtration of a Weyl module. Whenever M and N are finite rank free R modules, and $\phi : M \to N$ is an R homomorphism such that $\phi \otimes 1 : M_K \to N_K$ is bijective, we define $\nu_{\mathfrak{m}}(\det\phi)$ to be the length of the R module $\operatorname{coker}(\phi)$. If R bases m_i of M and n_i of N are picked so that $\phi(m_i) = a_i n_i$ for $a_i \in R$, then $\nu_{\mathfrak{m}}(\det\phi) = \sum_i \nu_{\mathfrak{m}}(a_i)$; this justifies the terminology.

Following Jantzen, we define the *Jantzen Filtration* of $\Delta_R(\lambda)$; the corresponding filtration of $\Delta(\lambda)$ is obtained by taking the canonical image of each term in the filtration of $\Delta_R(\lambda)$. For $i \geq 0$, put

$$\mathcal{J}^i\Delta_R(\lambda) = \{x \in \Delta_R(\lambda) \ : \ (x)c_\lambda \in \pi^i\nabla_R(\lambda)\} \tag{4.1.a}$$

Proposition 4.1.1. (a) *The filtration $\mathcal{J}^i$ is independent of the choice of c_λ.*

(b) *For each $i \geq 1$, $\mathcal{J}^i\Delta_R(\lambda)$ is an A_R submodule of $\Delta_R(\lambda)$ and $\mathcal{J}^i\Delta(\lambda)$ is an A submodule of $\Delta(\lambda)$.*

(c) $\Delta(\lambda)/\mathcal{J}^1\Delta(\lambda) \simeq L(\lambda)$.

(d) $\sum_{s>0} \dim_{\mathrm{k}} \mathcal{J}^s\Delta(\lambda) = \nu_{\mathfrak{m}}(\det(c_\lambda))$.

Proof. Any $c' : \Delta_R(\lambda) \to \nabla_R(\lambda)$ may be written as $c' = d \circ c_\lambda$ for some $d \in D_R^\lambda$; if c' yields a non-zero homomorphism mod $\mathfrak{m}$, then d does as well, hence d is a unit. This proves (a). Statement (b) is clear by construction. The head of $M = \Delta(\lambda)/\mathcal{J}^1\Delta(\lambda)$ is certainly $L(\lambda)$. Since $L(\lambda)$ occurs only once in a composition series for $\Delta(\lambda)$, (c) will follow provided we show $\operatorname{soc} M$ is $L(\lambda)$ isotypic. Let $L(\mu)$ be a simple submodule of $\operatorname{soc} M$, and let $\bar{v_\mu}$ be an A generator for this submodule. If v_μ denotes a lift of this vector to $\Delta(\lambda)$, then $v_\mu \notin \mathcal{J}^1\Delta_R(\lambda)$, so $c_\lambda(v_\mu) \neq 0$. Hence c_λ restricts to an inclusion $L(\mu) \to \operatorname{soc}\nabla(\lambda) = L(\lambda)$, and $\mu = \lambda$. Statement (d) follows by the argument in [Jan87, II.8.18]. ☐

4.2. The Andersen filtration. We will now define filtrations of certain spaces associated with tilting modules. Our discussion follows the work of Andersen [And97a, And97b]. Let Q_R be an admissible A_R lift of the partial tilting A module Q. Choose $\lambda \in \Omega$, and put

$$F_\lambda(Q_R) = \operatorname{Hom}_{A_R}(\Delta_R(\lambda), Q_R), \qquad \text{and } E_\lambda(Q_R) = \operatorname{Hom}_{A_R}(Q_R, \nabla_R(\lambda)). \tag{4.2.a}$$

Denote by $d_\lambda = \operatorname{rank}_R D_R^\lambda = \dim_k D^\lambda$. Q_R has a Δ_R filtration and a ∇_R filtration; it follows from (3.4.a) that $\operatorname{rank}_R F_\lambda(Q_R) = d_\lambda \cdot [Q_R : \Delta_R(\lambda)]_\Delta$ and $\operatorname{rank}_R F_\lambda(Q_R) = d_\lambda \cdot [Q_R : \nabla_R(\lambda)]_\nabla$. In particular, Proposition 3.4.2 shows that $F_\lambda(Q_R)$ and $E_\lambda(Q_R)$ have the same rank as R modules.

Let $F_\lambda(Q)$ and $E_\lambda(Q)$ be the reductions mod π of these R-modules; according to Proposition 3.3.2 $F_\lambda(Q)$ and $E_\lambda(Q)$ are isomorphic to $\operatorname{Hom}_A(\Delta(\lambda), Q)$ respectively $\operatorname{Hom}_A(Q, \nabla(\lambda))$.

For $\phi \in F_\lambda(Q_R)$ and $\psi \in E_\lambda(Q_R)$, we regard $\phi \circ \psi$ as an element of $\operatorname{Hom}_{A_R}(\Delta_R(\lambda), \nabla(\lambda))$; as this homomorphism space is a $D_R^\lambda - D_R^\lambda$ bi-module which is freely generated from either side by c_λ, we may write $\phi \circ \psi = c_\lambda d$ for a unique $d \in D_R^\lambda$. This gives a D_R^λ-valued bilinear pairing on $F_\lambda(Q_R) \times E_\lambda(Q_R)$; using this pairing, one defines a filtration of $F_\lambda(Q_R)$ by

$$\mathcal{A}^j F_\lambda(Q_R) = \{\phi \in F_\lambda(Q_R) : \phi \circ \psi \in c_\lambda \pi^j D_R^\lambda \ \forall\ \psi \in E_\lambda(Q_R)\}; \tag{4.2.b}$$

the corresponding filtration of $F_\lambda(Q)$ is obtained as above.

Lemma 4.2.1. *Let $\theta_\lambda : F_\lambda(Q_R) \to E_\lambda(Q_R)^*$ be given by*

$$\theta_\lambda(\phi) : \psi \longmapsto \phi \circ \psi = c_\lambda d \longmapsto tr_{D^\lambda_K/K}(d) \quad \text{for } \phi \in F_\lambda(Q_R), \psi \in E_\lambda(Q_R),$$

where $tr_{D^\lambda_K/K}$ is the reduced trace and d is the unique element in D^λ_R such that $\phi \circ \psi = c_\lambda d$. Define a filtration of $F_\lambda(Q_R)$ by

$$\mathcal{B}^j F_\lambda(Q_R) = \{\phi \in F_\lambda(Q_R) : \phi_\lambda(\phi) \in \pi^j E_\lambda(Q_R)^*\}.$$

Then $\mathcal{A}^j F_\lambda(Q_R) \subseteq \mathcal{B}^j F_\lambda(Q_R)$ for all j. If $[D^\lambda_K : K] = 1$, the filtrations coincide.

Proof. Let $\phi \in \mathcal{A}^j F_\lambda(Q_R)$. Then $\phi \circ \psi \in \pi^j D^\lambda_R$ for every $\psi \in E_\lambda(Q_R)$; for a fixed ψ, we have $\phi \circ \psi = c_\lambda \pi^j d$ where d is a unit in D^λ_R. Then $\theta_\lambda(\phi)(\psi) = \pi^j \mathrm{tr}_{D^\lambda_K/K}(d) \in \pi^j R$. Hence $\theta_\lambda(\phi) \in \pi^j E_\lambda(Q_R)^*$ which proves that $\mathcal{A}^j \subseteq \mathcal{B}^j$.

If $D^\lambda_K = K$, equality of the filtrations $\mathcal{A}^\cdot$ and $\mathcal{B}^\cdot$ is obvious. □

Proposition 4.2.2. *Let Q denote a tilting A module.*

(a) $\sum_{j>0} \dim_{\Bbbk} \mathcal{B}^j F_\lambda(Q) = \nu_{\mathfrak{m}}(\det \theta_\lambda)$

(b) $d_\lambda \cdot [Q_R : T_R(\lambda)]_T = \dim_{\Bbbk} F_\lambda(Q)/\mathcal{A}^1 F_\lambda(Q)$

Proof. (a) follows as for the Jantzen filtration; see [Jan87, II.8.18]. For (b), write

$$Q_R \simeq \bigoplus_{\lambda \in \Omega} T_R(\lambda)^{t_\lambda} \text{ with } t_\lambda \in \mathbb{Z}_{\geq 0};$$

then $[Q : T(\lambda)] = t_\lambda$, and we must show that $t_\lambda \cdot d_\lambda = \dim_{\Bbbk} F_\lambda(Q)/\mathcal{A}^1 F_\lambda(Q)$.

Consider a homomorphism $\phi : \Delta_R(\lambda) \to Q_R$; by construction, there is a map $\psi : Q_R \to \nabla_R(\lambda)$ such that $\phi \circ \psi$ is non-zero modulo π if and only if $\phi \notin \mathcal{A}^1(F_\lambda(Q_R))$.

If $t_\lambda = 0$, the result follows immediately from Proposition 2.1.5. If $t_\lambda > 0$, choose a summand $T_R(\lambda) \subset Q_R$, and choose corresponding ϕ and ψ so that $\phi \circ \psi$ is non-zero modulo π. Let $Q'_R = Q_R/T_R(\lambda)$. Then $F_\lambda(Q'_R) \simeq F_\lambda(Q_R)/\phi D^\lambda_R$ and the result follows by induction (compare [And97a, Proposition 1.5]). □

Assume now that the $(A, \Omega, \leq, \iota)$ is a QH algebra with duality, and that Q is a self-dual full tilting A module. If A_R possesses an anti-automorphism ι_R as in 3.5, we say that ι is defined over R. As described in section 3.5, $A'_R = \mathrm{End}_{A_R}(Q_R)$ has a corresponding anti-automorphism ι'_R.

Theorem 2. *Let A_R an R form of A, where $(A, \Omega, \leq, \iota)$ is a QH algebra with duality. Assume ι is defined over R.*

(a) *For each $\lambda \in \Omega$, there is an isomorphism $\gamma : F_\lambda(Q_R) \xrightarrow{\simeq} \Delta'_R((w\lambda)')$ of A'_R modules.*

(b) *For any choice of γ as in (a), $\gamma(\mathcal{B}^j) = \mathcal{J}^j$ for $j \geq 0$.*

(c) *If $\dim_K D^\lambda = 1$, $\gamma(\mathcal{A}^j) = \mathcal{J}^j$ for $j \geq 0$. Thus, the Andersen filtration and the Jantzen filtration may be identified.*

Proof. For (a), the existence of the isomorphism γ of A'_R modules follows from the corresponding assertion over A by uniqueness of A_R lifts of Δ modules as in 3.2; that there is an A' isomorphism $F_\lambda(Q) \to \Delta'((w\lambda'))$ follows from Lemma 2.3.2.

For (b), note first (again using uniqueness of lifts) that $E_\lambda(Q_R) \simeq \Delta'_R(\lambda')$ and thus Theorem 1 implies that $E_\lambda(Q_R)^\vee \simeq \nabla'_R(\lambda')$. The map θ_λ of Lemma 4.2.1 is A'_R linear when the R module $E_\lambda(Q_R)$ is regarded as the A'_R module $E_\lambda(Q_R)^\vee$. It suffices by Proposition 4.1.1(a) to show that the map $\gamma \circ \theta_\lambda$ is non-zero modulo $\mathfrak{m}$. This follows immediately from the fact, established in the proof of Theorem 1, that the "evaluation pairing" $\mathrm{tr} \circ \varepsilon$ is non-0.

Finally, (c) follows from (b) together with Lemma 4.2.1. □

4.3. Assume that A is a separable A module in the sense of [CR81, §7]; i.e. for each $\lambda \in \Omega$, $D^\lambda = \mathrm{End}_A(L(\lambda))$ is a separable k algebra for every $\lambda \in \Omega$. For example, this is always true when k is perfect, and in particular when k is finite. Choose a separable and finite field extension $k \subset \ell$ that splits all D^λ; i.e. so that $D^\lambda \otimes_k \ell$ is a split semisimple ℓ algebra. Let R_ℓ denote a complete discrete valuation ring which is an unramified extension of R with residue field ℓ, and fraction field $\mathcal{L}$; such an unramified R_ℓ exists say by [Ser79, Prop. 15]. If k is perfect of characteristic $p > 0$ and K has characteristic 0, there is even [Ser79, Theorem 3] a unique such R_ℓ unramified over R.

Let A_ℓ, A_{R_ℓ}, $A_\mathcal{L}$ be obtains from A by extension of scalars; it follows that for $F = \ell$ or L, the endomorphism algebra of each simple A_F module is again F. Since R_ℓ is unramified over R, the element π remains prime in R_ℓ. Furthermore, we claim that A_ℓ is again quasi-hereditary and $A_\mathcal{L}$ is again semisimple. Well, the latter claim is immediate say from [CR81, Corollary 7.8].

To see that A_ℓ is QH, let $\tilde{\Omega}$ parameterize the simple A_ℓ modules, and define $\varepsilon : \tilde{\Omega} \to \Omega$ by $\varepsilon(\mu) = \lambda$ provided $[L(\lambda) \otimes_k \ell : L_\ell(\mu)] \neq 0$; it follows from [CR81, Theorem 7.9] that ε is everywhere defined and single-valued on $\tilde{\Omega}$, and that $[L(\lambda) \otimes_k \ell : L_\ell(\mu)] \leq 1$ for all $\lambda \in \Omega$, $\mu \in \tilde{\Omega}$.

Since $P(\lambda) \otimes_k \ell$ is A_ℓ projective, it must be isomorphic to $\bigoplus_{\mu \in \varepsilon^{-1}(\lambda)} P_\ell(\mu)$ as A_ℓ modules. Define the partial order $\leq$ on $\tilde{\Omega}$ by "pull-back"; i.e. $\mu < \tau$ if and only if $\varepsilon(\mu) < \varepsilon(\tau)$; note that μ is incomparable to any τ with $\mu \neq \tau \in \varepsilon^{-1}(\varepsilon(\mu))$. With this partial order, it is straightforward to see that $\Delta_\ell(\lambda) \simeq \bigoplus_{\mu \in \varepsilon^{-1}(\lambda)} \Delta_\ell(\mu)$. Since $\leq$ is equivalent to any refinement of itself obtained by arbitrarily ordering each "fiber" $\varepsilon^{-1}(\varepsilon(\mu))$, it is straightforward to see that $\leq$ is equivalent to a total ordering. on $\tilde{\Omega}$. Since $[L(\lambda) \otimes_k \ell : L_\ell(\mu)] \leq 1$ for all $\lambda \in \Omega$, $\mu \in \tilde{\Omega}$, A_ℓ is QH. Thus, we have the

Proposition 4.3.1. *Let $(A, \Omega, \leq)$ be quasi-hereditary k algebra, and assume that the regular A module is separable. Let ℓ be a finite separable field extension of k. Then $(A_\ell, \tilde{\Omega}, \leq)$ is again quasi-hereditary.*

Furthermore, if ι is an antiautomorphism of A compatible with the order on Ω, then $\iota \otimes 1_\ell$ is compatible with the order on $\tilde{\Omega}$.

(The assertion regarding $\iota \otimes 1_\ell$ is straightforward to verify). We may therefore arrange, by extension of scalars, that the filtrations $\mathcal{A}^\cdot$ and $\mathcal{B}^\cdot$ are identical.

Remark 4.3.2. The separability hypothesis is necessary for the Proposition. Indeed, let $\ell \supset k$ be a purely inseparable extension field with $[\ell : k] = p = \mathrm{char}(k)$. Then the k algebra $A = \ell$ is QH. However, the ℓ algebra $A_\ell = \ell \otimes_k \ell$ has a unique simple module S with projective cover P_S such that $[P_S : S] = p$; hence A_ℓ is not QH.

5. Algebraic Groups

5.1. Suppose that k is a field of characteristic p, let $R = \mathbb{Z}_p$ be the (complete) ring of p-adic integers, and let $K = \mathbb{Q}_p$ denote the fraction field of R. The inclusion $\mathbb{F}_p \subset k$ makes k into an R module.

Let $G = G_k$ denote a connected reductive algebraic k-group scheme defined and split over $\mathbb{F}_p$, with maximal torus T contained in the Borel subgroup B. There is a split reductive R group scheme G_R from which G may be obtained via base change; for more details see 5.3 below. Denote G_K the K group scheme obtained by base change from G_R.

Write $X(T)$ for the characters on T; $X = X(T)$ contains the root system Φ. The choice of Borel subgroup determines a system of simple roots and positive roots $\Delta \subset \Phi^+ \subset \Phi$. Let X^+ denote the dominant weights in X.

For $\lambda \in X^+$, let $\nabla(\lambda) = H^0(G/B^-, \lambda)$ be the induced module as in [Jan87, I.3.3], and let $\Delta(\lambda) = \nabla(-w_0\lambda)^*$, where w_o is the longest word in the Weyl group. Finally, put $L(\lambda) = \operatorname{head}\Delta(\lambda) = \operatorname{soc}\nabla(\lambda)$; $L(\lambda)$ is the simple module with highest weight λ, and the $L(\lambda)$ are a complete set of rational simple G-modules.

The same constructions yield modules for G_K; for this group we have $\Delta_K(\lambda) = \nabla_K(\lambda) = L_K(\lambda)$ for all $\lambda \in X^+$. It is well known that the rational G_K modules are all semisimple.

We have by [Jan87, Prop II.2.8]

$$\operatorname{End}_G(L(\lambda)) \simeq k \text{ and } \operatorname{End}_{G_K}(L_K(\lambda)) \simeq K. \tag{5.1.a}$$

For a rational G module M, denote by $\operatorname{ch} M \in \mathbb{Z}[X]$ its character. For $\lambda \in X^+$, let $\chi(\lambda)$ be the character of $L_K(\lambda)$, the simple module in characteristic 0. The $\chi(\lambda)$ are given by the Weyl character formula, and we have as well that $\operatorname{ch}\Delta(\lambda) = \operatorname{ch}\nabla(\lambda) = \chi(\lambda)$.

5.2. A *saturated set of weights* is a finite subset $\pi \subset X$ so that π contains the root string $\lambda - r\alpha$ for $0 \le r \le \langle \lambda, \alpha^\vee \rangle$ whenever $\lambda \in \pi$. Equivalently, whenever $\lambda \in \pi^+ = \pi \cap X^+$, π contains the Weyl group orbit $W\mu$ for each dominant weight $\mu \le \lambda$.

A rational G_F module M will be said to belong to π provided each non-zero weight of M lies in π, for $F = k, K, R$. When $F = k, K$, this is the same as to say that each composition factor of M has highest weight in π^+.

For each saturated set of weights π, Donkin [Don86] constructs the generalized Schur algebras $\mathcal{S}_R(\pi)$ which are finitely generated and free over R. Let $\mathcal{S}(\pi) = \mathcal{S}_R(\pi) \otimes_R k$, and $\mathcal{S}_K(\pi) = \mathcal{S}_R(\pi) \otimes_R K$.

As before, we call a rational G_R module, or an $\mathcal{S}_R(\pi)$ module, admissible if it is a finite rank free R module. Donkin's construction yields: [Don86, §1,§3.2]

5.2.1. *For $F = k, K$, the category of $\mathcal{S}_F(\pi)$ modules is naturally equivalent to the category of rational G_F modules belonging to π. The category of admissible $\mathcal{S}_R(\pi)$ modules is naturally equivalent to the category of admissible rational G_R modules belonging to π.*

In particular, $\mathcal{S}_K(\pi)$ is a semisimple K algebra. Furthermore,

5.2.2. [Don93, §1] $\mathcal{S}(\pi)$ *is a quasi-hereditary algebra, where $\Omega = \pi^+$ with the usual partial ordering on dominant weights. The standard modules, respectively the Weyl modules, are precisely the $\nabla(\lambda)$, respectively $\Delta(\lambda)$, for $\lambda \in \Omega$ described above.*

The results in 3.2 show that for $\lambda \in X_+$, there are admissible G_R modules $\Delta_R(\lambda)$ and $\nabla_R(\lambda)$ that yield $\Delta(\lambda)$ and $\nabla(\lambda)$ on reduction mod p. Of course, it is well known that one can construct these modules as admissible R-lattices in the corresponding simple G_K module.

A tilting module T for G is a module which has both a Δ and a ∇ filtration. The equivalence of categories 5.2.1 yields for each dominant weight λ an indecomposable tilting G module $T(\lambda)$, and a corresponding admissible G_R module $T_R(\lambda)$.

Since $\operatorname{ch}\Delta(\lambda) = \operatorname{ch}\nabla(\lambda) = \chi(\lambda)$ for each $\lambda \in X^+$, the set of weights of either $\Delta(\lambda)$ or $\nabla(\lambda)$ is a saturated set of weights by [Hum80, Prop 21.3]. The union of finitely many saturated sets of weights is again saturated; it follows that the set of weights $\Pi(T)$ of a tilting module T is saturated; in particular, we may speak of the Schur algebra $\mathcal{S}(\Pi(T))$. We will say that T is a *full tilting G module* provided T is a full tilting module for $\mathcal{S}(\Pi(T))$. We summarize with:

5.2.3. *Let T be a tilting G module. Then the Schur algebra $\mathcal{S}(\Pi(T))$ is defined, and the equivalence of categories 5.2.1 gives a canonical $\mathcal{S}(\Pi(T))$ module structure on T. Furthermore,*

$$\operatorname{End}_G(T) \simeq \operatorname{End}_{\mathcal{S}(\Pi(T))}(T).$$

If T is a full tilting G module (i.e. a full tilting $\mathcal{S}(\Pi(T))$ module), then $\operatorname{End}_G(T) \simeq \mathcal{S}(\Pi(T))'$ *is the Ringel dual of* $\mathcal{S}(\Pi(T))$ *with respect to T.*

Remark 5.2.4. The usual Jantzen filtration [Jan87, II.8] of $\Delta(\lambda)$ and the Andersen filtration of $F_\lambda(T)$ [And97a], for a tilting module T, coincide with the filtrations $\mathcal{J}$ and $\mathcal{A}$ defined here with respect to any generalized Schur algebra $\mathcal{S}(\pi)$ for which $\Delta(\lambda)$, respectively T, belong to π. In view of (5.1.a), the filtrations $\mathcal{A}$ and $\mathcal{B}$ for $F_\lambda(T)$ coincide.

An anti-involution τ of G_R as in [Jan87, II.1.16] is called a *Chevalley involution*; each reductive group possesses a Chevalley involution (*loc. cit.)* For a G_R module M, let $M^{\vee}$ denote the "contravariant dual" module; as an R module $M^{\vee} = M^*$ is the usual R-dual. The G_R action on $M^{\vee}$ is defined by twisting with the Chevalley involution τ. This contravariant dual satisfies:

$$\Delta_R(\lambda)^{\vee} \simeq \nabla_R(\lambda) \tag{5.2.a}$$

for all $\lambda \in X^+$; see [Jan87, II.2.13] where this is observed for the modules $\Delta_k(\lambda)$ and $\nabla_k(\lambda)$. The discussion in 3.5 shows that (5.2.a) holds over R. One now has the following:

5.2.5. *For any saturated set of weights π, there is an antiautomorphism $\iota : \mathcal{S}_R(\pi) \to \mathcal{S}_R(\pi)$ for which the duality $M \mapsto M^{\vee}$ arises via twisting with ι. Each tilting module and each simple module for $\mathcal{S}(\pi)$ is self dual. The data $(\mathcal{S}(\pi), \Omega, \leq, \iota)$ form a QH algebra with duality.*

Proof. This is straightforward from the construction given in [Don86, §3.2]; since $\mathcal{S}_R(\pi)$ is defined as a quotient of Kostant's R form of the enveloping algebra of the corresponding semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, one must verify that ι leaves invariant the defining ideal. This fact is trivial from the definition.

The assertion of self-duality of tilting modules follows once it is known for indecomposable tilting modules; in that case, the assertion follows easily from (5.2.a). In view of Lemma 2.3.2, this proves that the permutation w of Ω induced by the duality is the identity; in particular, ι is compatible with the order on Ω. □

5.3. R-forms. If G_R is an R group scheme, and B is any (commutative) R algebra, the notation G_B denotes the group scheme obtained by base change, and $G_R(B)$ denotes the group of B points of G_R. We say that a connected affine R group scheme G is *split reductive* provided $R[G]$ is free over R, and that for each R-algebra L that is a field, G_L is a split reductive algebraic group defined over L in the sense, say, of [Bor91]. In particular, G_k and G_K are split reductive, where k is the residue field $\mathbb{F}_p$ and K is $\mathbb{Q}_p$.

Split reductive groups are classified by their root datum, and it is known that there are split reductive algebraic groups $G_{\mathbb{Z}}$ for every root datum; see [Bor70]. In this section we give explicit constructions for the corresponding split reductive R group schemes in a few cases. We will say that G_R is almost simple, simply connected, etc. provided these descriptions are valid for the root system of G_R.

Suppose that G is an R group scheme. If T is a torus in G, and M is a finite, free R module which is a rational representation of G, then T acts diagonally on M with respect to some R basis e_i; these basis vectors determine characters $\varepsilon_i \in X(T)$.

5.3.A. *General Linear Case.* Let V_R denote a free R module of rank r for some integer $r \geq 2$. Let $G_R = \mathrm{GL}(V_R)$ denote the general linear R group scheme. In [Jan87, II.1.21], it is shown that this is a simply connected, connected, reductive R group scheme of type A_{r-1} with a 1 dimensional central torus. As in *loc. cit.*, let T_R be the maximal torus of G_R acting diagonally with respect to a basis e_i of V_R; the corresponding characters ε_i form a $\mathbb{Z}$ basis for X. For $1 \leq j \leq r$, put $\varpi_j = \varepsilon_1 + \cdots + \varepsilon_j \in X$; for $1 \leq j \leq r-1$, the characters ϖ_j are lifts to X of the fundamental dominant weights for $\mathrm{SL}(V_R)$.

Let $G'_R = \mathrm{SL}(V_R)$. When $r = 2$, the group G'_R has semisimple rank 1. In this case, $G'_R = \mathrm{SL}_{2,R}$ is a connected, simply connected, quasisimple group scheme of type A_1; the weight lattice may be identified with $\mathbb{Z}$ via $\varpi_1 \longmapsto 1$, where ϖ_1 denotes the fundamental dominant weight.

5.3.B. *Symplectic Case.* Let V_R denote a free R module of rank $2r$, $r \geq 1$, and let α be an alternating R-valued form on V_R. (V_R, α) is an R-split symplectic space provided that there are *admissible totally isotropic* (ATI) R-subspaces $E_R, F_R \subset V_R$; this means that E_R and F_R are isotropic, that $V_R = E_R \oplus F_R$, and that the restriction of α to $E_R \times F_R$ is a perfect pairing.

Let (V_R, α) be R-split, and let $G_R = \mathrm{Sp}(V_R)$ denote the R subgroup scheme of $\mathrm{GL}(V_R)$ which leaves stable the form. By [Bor91, V.23.3, V.23.4], G_R is a simply connected split almost simple R group scheme of type C_r.

A maximal torus of G_R may be chosen as the subgroup scheme of a suitable maximal torus of $\mathrm{GL}(V_R)$ stabilizing a pair of ATI subspaces E_R, F_R (*loc. cit.*). If $\{e_i\} \subset E_R$, $\{f_j\} \subset F_R$ are dual bases, the characters ε_i dual to the basis vectors e_i form a $\mathbb{Z}$ basis of X, and the fundamental dominant weights are $\varpi_j = \varepsilon_1 + \cdots + \varepsilon_j$.

When $r = 1$, the form on V_R coincides with the determinant pairing $V_R \times V_R \to \bigwedge^2 V_R = Re_1 \wedge e_2$ hence G_R coincides with the group scheme $\mathrm{SL}(V_R)$.

5.3.C. *Spin groups.* Let V_R be a free R module of rank $2r + \varepsilon$ with $\varepsilon \in \{0, 1\}$, and let β be a symmetric R-valued form on V_R. Let $q(v) = \frac{1}{2}\beta(v, v)$. When $\varepsilon = 0$, (V_R, q) (or (V_R, β)) is an R-split orthogonal space provided there are *admissible totally singular* (ATS) subspaces $E_R, F_R \subset V_R$; this means that E_R, F_R are singular, that $V_R = E_R \oplus F_R$, and that the restriction of α to $E_R \times F_R$ is a perfect pairing. When $\varepsilon = 1$, (V_R, β) is R-split provided that $V_R = W_R \oplus L_R$ is an orthogonal sum where W_R is R-split as above, $L_R = Rv$ has rank 1, and $q(v) \in R^\times$ is a unit.

Let $\mathrm{Cl}(q)$ denote the $\mathbb{Z}/2\mathbb{Z}$ graded Clifford algebra of q (see [KMRT98, §8] for the definition), and let $G_R = \mathrm{Spin}(q, V_R) = \mathrm{Spin}(q)$; this group scheme is defined in [KMRT98, VI §23] as a subgroup scheme of the unit group scheme of $\mathrm{Cl}^0(q)$ (even part). G_R is a simply connected, almost simple R group scheme of type D_r if $\varepsilon = 0$, and of type B_r otherwise (*loc. cit.* VI §25 (25.10), (25.12)).

Let E_R, F_R be ATS subspaces of V_R, let $\{f_i\}$ denote an R basis for F, and let $f = f_1 \cdots f_r \in \mathrm{Cl}(q)$; then $\mathbb{S}_R(F) = \mathrm{Cl}(q)f$ is a left ideal and hence affords a representation for the group scheme G_R known as the spin module. Different choices of E, F lead to isomorphic G_R representations, so the notation $\mathbb{S}_R$ is unambiguous.

There is a unique R-algebra map $\Psi : \bigwedge^{\cdot} E_R \to \mathrm{Cl}(q)$ such that $\Psi\,|_{E_R}$ is the inclusion of E_R; basis considerations show that Ψ is injective, and that $\mathrm{Cl}(q)$ is a free $\Lambda = \bigwedge^{\cdot} E_R$ algebra of rank $2^{r+\varepsilon}$. The ideal $\mathbb{S}_R(F)$ coincides with the free rank 1 left Λ submodule generated by f; thus we may identify $\mathbb{S}_R(F)$ with the exterior algebra on E_R. The R algebra Λ is $\mathbb{Z}/2\mathbb{Z}$ graded in a natural way hence so is $\mathbb{S}_R(F')$. When $\varepsilon = 0$, G_R preserves the graded components $\mathbb{HS}_R^{\pm}(F)$; these G_R modules are known as the half-spin modules.

Remark 5.3.1. Arguing as in [KMRT98, §8 Prop. 8.3], one can show that $\mathrm{CL}^0(q)$ is $\mathrm{End}_R(\mathbb{S}_R)$ when $\varepsilon = 1$ and is $\prod_{s \in \{\pm\}} \mathrm{End}_R(\mathbb{HS}_R^s)$ when $\varepsilon = 0$. It follows that $\mathrm{Spin}(q)$ is a closed subgroup scheme of $\mathrm{GL}(\mathbb{S}_R)$.

Let $\{e_i\} \subset E_R$, $\{f_j\} \subset F_R$ be dual bases; there is a unique maximal torus of $\mathrm{Spin}(q)$ which acts diagonally on the e_i,f_j; the characters ε_i determined by the basis elements e_i form a $\mathbb{Z}$ basis for the the character group X. In this basis, the fundamental dominant weights are given by the formulae in [Bou72, Ch. 4-6, p. 256,Planche IV].

It is well known that when V_R has rank 3, $\mathrm{Spin}(q)$ is the group scheme $\mathrm{SL}(\mathbb{S}_R) = \mathrm{SL}_2$. (This follows from the isomorphism theorem for split reductive groups, but can be seen immediately from the construction in this case.)

5.3.D. *Morphisms over R.* In this section we will describe certain morphism between the R group schemes constructed above. It is likely that the morphisms could be obtained by invoking [DG70, Théorème 4.1, Exposé XXIII] and simple connectivity arguments; however, we give instead constructions arising from the explicit realizations.

If $V_R \simeq W_R \oplus W_R'$ are finite rank free R modules, then there is an obvious closed embedding of $\mathrm{GL}(W_R)$ in $\mathrm{GL}(V_R)$ (for any action of $\mathrm{GL}(W_R)$ on W_R'). This has the following consequences:

5.3.2. *Let V_R, W_R be finite rank free R modules, and let $X_R = V_R \otimes_R W_R$. There is a closed embedding of $G = \mathrm{GL}(V_R)$ in $\Gamma = \mathrm{GL}(X_R)$ (with trivial G action on W_R). Moreover, there is an isomorphism $res_G^{\Gamma}(\bigwedge^{\cdot} X_R) \simeq \bigwedge^{\cdot}(\oplus_t V_R)$ where t is the rank of W_R.*

5.3.3. *Let V_R, α be an R-split symplectic space. There are closed embeddings of* $\mathrm{Sp}(V_R)$ *in* $\mathrm{GL}(V_R \otimes_R W_R)$ *for any finite free R module W_R (with trivial* $\mathrm{Sp}(V_R)$ *action on W_R), and in* $\mathrm{GL}(\bigwedge^{\cdot} V_R)$ *(with the usual action on the exterior powers).*

Let (V_R, q) be an R-split orthogonal space; using Remark 5.3.1 one easily proves

5.3.4. *Suppose that G_R is a closed subgroup scheme of* $\mathrm{SO}(V_R, q)$ *which leaves stable a pair of ATS spaces $E_R, F_R \subset V_R$. Let $\pi : \mathrm{Spin}(q) \to \mathrm{SO}(V_R, q)$ be the central isogeny as in* [KMRT98, §23.4]*, and let $\tilde{G_R} = \pi^{-1}(G_R)$. Then the action of G_R on $\bigwedge^{\cdot} E_R = \mathbb{S}_R(F_R)$ defines a morphism $G_R \to \tilde{G_R} \to \mathrm{Spin}(q)$ over R.*

Let (V_R, α) and (W_R, β) be two R-split symplectic spaces; suppose that V_R has rank $2r$ and W_R has rank $2t$. The product form $\alpha \otimes \beta$ on $X_R = V_R \otimes_R W_R$ is symmetric, and (X_R, q) is an R-split orthogonal space, where q is the corresponding quadratic form.

Let (E_R, F_R) and (E'_R, F'_R) be ATI subspaces of V_R and W_R respectively (see 5.3.B). The subspaces $A_R = V_R \otimes E'_R, B_R = V_R \otimes F'_R$ of X_R are ATS, and $G = \mathrm{Sp}(V_R)$ evidently leaves stable the pair A_R, B_R. According to 5.3.3 there are closed embeddings of G in $\mathrm{SO}(X_R)$ and in $\mathrm{GL}(\mathbb{S}_R(B_R)))$; thus 5.3.4 together with Remark 5.3.1 yield:

5.3.5. *There is a closed embedding of G in $\Gamma = \mathrm{Spin}(q)$. Moreover, $res^{\Gamma}_G(\mathbb{S}(B_R)) \simeq \bigwedge^{\cdot}(\oplus_t V_R)$.*

Let (V_R, q) and (W_R, q') be R-split orthogonal spaces; suppose that V_R has rank $2r+1$ and W_R has rank $2t+1$. Then $(X_R = V_R \otimes_R W_R, Q = q \otimes q')$ is again an R-split orthogonal space.

Let (E_R, F_R) and (E'_R, F'_R) be ATS subspaces respectively of $V_R = E_R \oplus F_R \oplus Rv$ and $W_R = E'_R \oplus F'_R \oplus Rw$ (see 5.3.C). The universal property of Clifford algebras gives an R-algebra homomorphism $\mathrm{Cl}(q) \to \mathrm{Cl}(Q)$ corresponding to the subspace $V_R \otimes w$ of X_R; basis considerations show this to be injective and R split. This homomorphism on Clifford algebras induces a corresponding group scheme morphism. The subspaces $A_R = (V_R \otimes_R E'_R) \oplus (E_R \otimes w)$, $B_R = (V_R \otimes_R F'_R) \oplus (u \otimes E'_R)$ of X_R are ATS; for $G = \mathrm{Spin}(q)$ this yields:

5.3.6. *There is a closed embedding of G in $\Gamma = \mathrm{Spin}(Q)$. Let $\mathbb{S}^{\Gamma}_R$ denote the spin module for Γ, and $\mathbb{S}^G_R$ that for G. Then $res^{\Gamma}_G(\mathbb{S}^{\Gamma}_R(B_R)) \simeq \bigwedge^{\cdot}(\oplus_t V_R) \otimes_R \mathbb{S}^G_R(F_R)$,*

6. Applications to dual pairs

6.1. Howe Duality. In [AR96], a positive characteristic analogue of R. Howe's notion of a dual reductive pair is introduced. This notion may be described as follows:

Let Γ denote a split connected reductive algebraic k group, and suppose G and H are split connected reductive subgroups of Γ. Let $\mathbf{M}$ be a rational Γ module, and finally let X_G and X_H denote the respective weight lattices of G and H. The data $(G, H, \Gamma, \mathbf{M})$ constitute a *Howe dual datum* if the following four conditions HD1-HD4 hold:

HD1 $G \subset C_\Gamma(H)$ and $H \subset C_\Gamma(G)$, where $C_\Gamma(X)$ denotes the centralizer in Γ of X.
HD2 $\mathrm{res}^{\Gamma}_G(\mathbf{M})$ is a full tilting G module and $\mathrm{res}^{\Gamma}_H(\mathbf{M})$ is a full tilting H module.

Let $\pi_G = \Pi(\mathrm{res}^{\Gamma}_G(\mathbf{M})) \subset X_G$ be the weights of the G module $\mathrm{res}^{\Gamma}_G(\mathbf{M})$; define $\pi_H \subset X_H$ analogously. In view of HD2 and 5.2.3, $\mathbf{M}$ is canonically a module for $\mathcal{S}(\pi_G)$ and $\mathcal{S}(\pi_H)$. HD1 implies that the natural module actions induce algebra homomorphisms

$$\gamma_H : \mathcal{S}(\pi_H) \to \mathrm{End}_G(\mathbf{M})^{\mathrm{op}} = (\mathcal{S}(\pi_G)')^{\mathrm{op}} \text{ and } \gamma_G : \mathcal{S}(\pi_G) \to \mathrm{End}_H(\mathbf{M})^{\mathrm{op}} = (\mathcal{S}(\pi_H)')^{\mathrm{op}},$$

where the Ringel duals $\mathcal{S}(\pi_G)'$ and $\mathcal{S}(\pi_H)'$ are both constructed using $\mathbf{M}$.

HD3 The homomorphisms γ_H and γ_G are bijective.

Let τ_G, τ_H be Chevalley involutions of G and H, and denote by ι_G, ι_H the corresponding antiautomorphisms of $\mathcal{S}(\pi_G)$ and $\mathcal{S}(\pi_H)$. Then

$$\tilde{\gamma}_G = \gamma_G \circ \iota_G : \mathcal{S}(\pi_G) \to \mathcal{S}(\pi_H)' \quad \text{and} \quad \tilde{\gamma}_H = \gamma_H \circ \iota_H : \mathcal{S}(\pi_H) \to \mathcal{S}(\pi_G)'$$

are algebra isomorphisms. Define bijections $\theta_G : \pi_G^+ \to \pi_H^+$ and $\theta_H : \pi_H^+ \to \pi_G^+$ by

$$\mathcal{F}_{\tilde{\gamma}_H}(L_{\mathcal{S}(\pi_G)'}(\lambda')) = L_{\mathcal{S}(\pi_H)}(\theta_G(\lambda)) \quad \text{and} \quad \mathcal{F}_{\tilde{\gamma}_G}(L_{\mathcal{S}(\pi_H)'}(\mu')) = L_{\mathcal{S}(\pi_G)}(\theta_H(\mu)), \tag{6.1.a}$$

with $\mathcal{F}$ as in 2.2.

HD4 θ_G and θ_H are inverse set bijections, and are anti-isomorphisms of partially ordered sets.

Remark 6.1.1. Even in the absence of HD3, HD2 implies that γ_H and γ_G are injective since a full tilting module is faithful by Proposition 2.1.3.

Observe that the notion of a Howe dual datum is symmetric in G and H; any preference given to one of the groups in the sequel is strictly for convenience.

Remark 6.1.2. Since the Chevalley involutions induce the identity maps on π_G^+ and π_H^+, HD4 implies that there is an isomorphism of QH algebras with duality

$$(\mathcal{S}(\pi_H), \pi_H^+, \leq, \iota_H) \simeq (\mathcal{S}(\pi_G)', (\pi_G^+)', \leq, \iota_G')$$

as in Theorem 1; thus we may identify π_H^+ with $(\pi_G^+)'$ (the opposite poset) via an order reversing bijection $' : \pi_G^+ \to \pi_H^+$ such that $F\nabla_G(\lambda) \simeq \Delta_H(\lambda')$.

Let $(G, H, \Gamma, \mathbf{M})$ be a Howe dual datum.

Proposition 6.1.3. *Let $\lambda, \mu \in \pi_G^+$. Then* $[T_G(\lambda) : \Delta_G(\mu)]_\Delta = [\Delta_H(\mu') : L_H(\lambda')]$

Proof. In view of the preceding Remark this follows from Proposition 2.1.10 combined with Proposition 3.4.2. □

Proposition 6.1.4. *Let T_H denote the maximal torus of H, and let $\mathrm{res}^\Gamma_{T_H}\mathbf{M} \simeq \bigoplus_{\mu \in \pi_H} M(\mu)$ be the weight space decomposition as a T_H module. Then each $M(\mu)$ is a G summand of $\mathrm{res}^\Gamma_G \mathbf{M}$; in particular, each $M(\mu)$ is a tilting module for G.*

Proof. Since T_H centralizes G, G preserves the weight spaces of T_H; that each $M(\mu)$ is a tilting module for G is immediate since the category of tilting modules is closed under direct summands. □

We say that the datum $(G, H, \Gamma, \mathbf{M})$ is *defined over R* provided there are split reductive R-group schemes G_R, H_R, Γ_R, where G_R and H_R are subgroup schemes of Γ_R, together with a rational Γ_R module $\mathbf{M}_R$, from which the corresponding datum over k arises via base change.

Theorem 3. *Suppose $(G, H, \Gamma, \mathbf{M})$ is a Howe dual datum defined over R. Let $\lambda \in \pi_G^+$.*

(a) *The H modules $F_{\lambda, G_R}(\mathbf{M}_R)$ and $\Delta_{G_R}(\lambda')$ are isomorphic.*

(b) *Let γ be any H-isomorphism $F_{\lambda, G_R}(\mathbf{M}_R) \xrightarrow{\simeq} \Delta_{G_R}(\lambda')$ as in (a). Then $\gamma(\mathcal{A}^i) = \mathcal{J}^i$ for $i \geq 0$; i.e. γ identifies the Andersen filtration and the Jantzen filtration.*

Proof. This follows immediately from Remark 6.1.2 and Theorem 2. (Note by Remark 5.2.4 the filtrations $\mathcal{A}$ and $\mathcal{B}$ coincide.) □

Theorem 4. *Let the datum $(G,H,\Gamma,\mathbf{M})$ be defined over R, and suppose HD1 and HD2 hold.*

(1) *If $\gamma_{G_K} : \mathcal{S}_K(\pi_G) \to \mathrm{End}_{H_K}(\mathbf{M}_K)^{op}$ and the analogous map γ_{H_K} are both K-algebra isomorphisms, then HD3 holds.*

(2) *If $\theta_{G_K} : \pi_G^+ \to \pi_H^+$ and $\theta_{H_K} : \pi_H^+ \to \pi_G^+$ satisfy the condition in HD4, where θ_{G_K} and θ_{H_K} are defined as in* (6.1.a)*, then HD4 holds for θ_H and θ_G.*

In particular, if the conditions in 1 and 2 hold, $(G,H,\Gamma,\mathbf{M})$ is a Howe dual datum.

Proof. For 1, note first that $\mathrm{End}_{H_R}(\mathbf{M}_R) \otimes_R k \simeq \mathrm{End}_H(\mathbf{M})$ by Proposition 3.3.2. Hence, condition 1 yields

$$\dim_k \mathrm{End}_H(\mathbf{M}) = \dim_K \mathrm{End}_{H_K}(\mathbf{M}_K) = \dim_K \mathcal{S}_K(\pi_G) = \dim_k \mathcal{S}(\pi_G).$$

Since γ_G is injective by Remark 6.1.1, equality of dimensions shows that γ_G is an isomorphism; similarly, γ_H is an isomorphism.

As to 2, note that $\nabla_{G_R}(\lambda) \otimes_R K = L_{G_K}(\lambda)$ for $\lambda \in \pi_G^+$. Since $\tilde{\gamma}_G$ and $\tilde{\gamma}_{G_K}$ arise via base change from a map $\tilde{\gamma}_{G_R}$, and since $\mathrm{soc}\,\nabla_{G_R}(\lambda) \otimes_R k = L_G(\lambda)$, $\theta_G = \theta_{G_K}$ as maps from $\pi_G^+ \to \pi_H^+$. Similar considerations handle θ_H. □

Remark 6.1.5. Compare part 1 of the Theorem with [AR96, Prop 4.4]. The result is not trivial; it isn't generally true that $\mathrm{End}_{B_R}(M_R) \otimes_R k \simeq \mathrm{End}_B(M)$ for arbitrary admissible R algebras B_R and admissible B_R modules M_R.

Remark 6.1.6. Condition 1 of Theorem 4 implies that $\mathcal{S}_K(\pi_G)$ and $\mathcal{S}_K(\pi_H)$ are Morita equivalent (see [CR81, §3D]). In characteristic 0, Howe [How89] described pairs of reductive groups G_K and H_K for which the hypotheses of Theorem 4 hold; these correspond to the "Howe dual pairs" considered in [Don93] and [AR96]. If one knows the datum to be defined over R, the Theorem should simplify the verification of the Howe duality properties: one only need check HD3 and HD4 "in characteristic 0". In characteristic p, one must verify that $\mathbf{M}$ is a full tilting module for G and for H. The main tool required for this is the *tensor product theorem*; see [Don85] and [Mat90].

6.2. Applications of Howe duality. Suppose that $(G,H,\Gamma,\mathbf{M})$ is a Howe dual datum. In this section, we consider the situation when H has semisimple rank 1. This situation is interesting because in the rank 1 case, the composition factors of the Weyl modules are known, the characters of the indecomposable tilting modules are known, and the Jantzen filtration of a Weyl module has an explicit description.

The weight lattice of $\mathrm{SL}_{2,k}$ may be identified with $\mathbb{Z}$; this being done, the dominant weights correspond to $\mathbb{Z}_{\geq 0}$. Similarly, the weight lattice of the group $\mathrm{GL}_{2,k}$ may be identified with $\mathbb{Z} \oplus \mathbb{Z}$; for suitable choices, we may and shall assume that the inclusion of the maximal torus of $\mathrm{SL}_{2,k}$ into a maximal torus for $\mathrm{GL}_{2,k}$ induces the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ via $(a,b) \mapsto a-b$ on weight lattices. Thus, $(a,b) \leq (a',b')$ if and only if $a+b = a'+b'$ and $a-b \leq a'-b'$; in particular, a weight $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ is dominant just in case $a \geq b$.

The indecomposable tilting modules for the group $\mathrm{SL}_{2,k}$ have been completely described by S. Donkin. In [Don93, §2, Example 2], Donkin gives an inductive tensor product description of these tilting modules. In [Erd94, 6.3], Erdmann gives a demonstration, due to Donkin, that the inductive description of these tilting modules leads to explicit formulae for the multiplicities

$$t(k,s) = [T(k) : \Delta(s)]_\Delta \tag{6.2.a}$$

for $\mathrm{SL}_{2,k}$. The corresponding formulas for $\mathrm{GL}_{2,k}$ are given by:

$$[T(k,l):\Delta(s,t)]_\Delta = \delta_{s+t,k+l}\cdot t(k-l,s-t) \tag{6.2.b}$$

The multiplicities $t(k,s)$ are "the same" as the decomposition matrix for the modular 2-part partition representations of the symmetric group obtained by James [Jam78, Theorem 24.15].

The Carter-Cline Theorem [CC75, Theorem 1.2] describes the composition factors of the Weyl modules for $\mathrm{SL}_{2,k}$; namely, it computes the multiplicities

$$w(k,s) = [\Delta(k):L(s)] \tag{6.2.c}$$

for $\mathrm{SL}_{2,k}$. The translation for $\mathrm{GL}_{2,k}$ reads:

$$[\Delta(k,l):L(s,t)] = \delta_{k+l,s+t}\cdot w(k-l,s-t) \tag{6.2.d}$$

We consider various possibilities for the rank 1 group H and for the saturated set of weights π_H. Since $(G,H,\Gamma,\mathbf{M})$ is assumed to be a Howe dual datum, the condition HD4 gives an order reversing bijection $' : \pi_G^+ \to \pi_H^+$ (see Remark 6.1.2). A convenient description of $'$ and of π_G^+ will be chosen. Throughout, S_R denotes a maximal torus of H_R.

For fixed $r\in\mathbb{Z}_{\geq 1}$, we consider the following three cases:

(6.2.e)

Case	H	Weights	$' : \pi_G^+ \to \pi_H^+$
I	$\mathrm{GL}_{2,k}$	$\pi_H^+ = \{(a,b) \mid 0\leq a\leq b\leq r\}$	
		$\pi_G^+ = \{\lambda(a,b) \mid 0\leq a\leq b\leq r\}$	$\lambda(a,b)' = (r-a,r-b)$
II	$\mathrm{SL}_{2,k}$	$\pi_H^+ = \{a \mid 0\leq a\leq r\}$	
		$\pi_G^+ = \{\lambda(a) \mid 0\leq a\leq r\}$	$\lambda(a)' = r-a$
III	$\mathrm{SL}_{2,k}$	$\pi_H^+ = \{2a+1 \mid 0\leq a\leq r\}$	
		$\pi_G^+ = \{\lambda(a) \mid 0\leq a\leq r\}$	$\lambda(a)' = 2(r-a)+1$

where a and b always denote integers.

Proposition 6.1.3 together with (6.2.a), (6.2.b), (6.2.c), and (6.2.d) yield:

Proposition 6.2.1. *Consider case I of* (6.2.e). *For $\lambda(i,j),\lambda(a,b)\in\pi_G^+$ one has*

(a) $[\Delta_G(\lambda(i,j)):L_G(\lambda(a,b))] = \delta_{i+j,a+b}\cdot t(b-a,j-i)$*, and*
(b) $[T_G(\lambda(i,j)):\Delta_G(\lambda(a,b))]_\Delta = \delta_{i+j,a+b}\cdot w(b-a,j-i)$.

Proposition 6.2.2. *Consider case II of* (6.2.e). *For $\lambda(i),\lambda(a)\in\pi_G^+$ one has*

(a) $[\Delta_G(\lambda(i)):L_G(\lambda(a))] = t(r-a,r-i)$*, and*
(b) $[T_G(\lambda(i)):\Delta_G(\lambda(a))]_\Delta = w(r-a,r-i)$.

Proposition 6.2.3. *Consider case III of* (6.2.e). *For $\lambda(i),\lambda(a)\in\pi_G^+$ one has*

(a) $[\Delta_G(\lambda(i)):L_G(\lambda(a))] = t(2(r-a)+1,2(r-i)+1)$*, and*
(b) $[T_G(\lambda(i)):\Delta_G(\lambda(a))]_\Delta = w(2(r-a)+1,2(r-i)+1)$.

Remark 6.2.4. Of course, the significance of these Propositions is that there are explicit recipes for determining the multiplicities. This provides a means for computing $\dim_k L_G(\lambda)$ for $\lambda\in\pi_G^+$.

Proposition 6.2.5. *Consider case I of* (6.2.e). *For* $(s,t) \in \pi_H^+$, *let*

$$Y(s,t) = Y_I(s,t) = \left\{ 0 \le e \le min(s, r-t) : p \not| \binom{t-s+2e}{e} \right\}.$$

There is a G_R *module isomorphism* $\mathbf{M}_R(s,t) \simeq \bigoplus_{e \in Y(s,t)} T_R(\lambda(s-e, t+e))$.

Proposition 6.2.6. *Consider case II or case III of* (6.2.e). *For* $s \in \pi_H^+$, *let*

$$Y(s) = Y_{II}(s) = \left\{ 0 \le a \le s : s \equiv a \pmod 2 \text{ and } p \not| \binom{r-a}{(s-a)/2} \right\}$$

in case II, or

$$Y(s) = Y_{III}(s) = \left\{ 0 \le a \le i \ : \ p \not| \binom{2(r-a)+1}{2r-s-a+1} \right\}$$

in case III. There is a G_R *module isomorphism.* $\mathbf{M}_R(s) \simeq \bigoplus_{a \in Y(s)} T_R(\lambda(a))$.

We provide a proof of 6.2.5; the proof of 6.2.6 is similar.

Proof of Prop. 6.2.5. Let $\lambda = \lambda(a,b) \in \pi_G^+$, and write $c = a+b$. Consider as well the corresponding weight $\gamma = \lambda(a,b)' = (r-b, r-a) \in \pi_H^+$. According to Theorem 3, one has $F_\lambda(\mathbf{M}_R) \simeq \Delta_{H_R}(\gamma)$. This implies that

$$F_\lambda(\mathbf{M}_R) \simeq \bigoplus_{i=0}^{c} \mathrm{Hom}_{G_R}(\Delta_{G_R}(\lambda), \mathbf{M}_R(i, c-i)); \tag{6.2.f}$$

indeed, the right hand side is precisely the space on which the central torus of G_R acts by the character (c,c). One knows that $\mathrm{rank}_R \Delta_{H_R}(\gamma) = c - 2a + 1$ and that each S_R weight space of $\Delta_{H_R}(\gamma)$ is free of rank 1 over R. Since $\mathrm{Hom}_{G_R}(\Delta_{G_R}(\lambda), \mathbf{M}_R(s,t))$ is the (s,t) weight space of $F_\lambda(\mathbf{M}_R)$ as an S_R module, it follows that $\mathrm{Hom}_{G_R}(\Delta_{G_R}(\lambda), \mathbf{M}_R(i, c-i))$ is non-0 if and only if $-c + 2a \le 2i - c \le c - 2a$, equivalently $a \le i \le c - a$.

For $0 \le k \le c - 2a$, let $\phi_k^\lambda : \Delta_{G_R}(\lambda(a,b)) \to \mathbf{M}_R(a+k, c-a-k)$ be an R generator for the $(a+k, c-a-k)$ weight space of $F_\lambda(\mathbf{M}_R)$; then $\{\phi_0^\lambda, \phi_1^\lambda \dots, \phi_{c-2a}^\lambda\}$ form an R-basis for $F_\lambda(\mathbf{M}_R)$.

Let $\mathcal{J}^1 = \mathcal{J}^1 \Delta_{H_R}(\gamma)$ denote the first proper term of the Jantzen filtration; it is well known that $\phi_i^\lambda \in \mathcal{J}^1$ if and only if p divides $\binom{c-2a}{i}$. Theorem 3 shows that $\mathcal{J}^1$ may be identified with $\mathcal{A}^1$, the first proper term of Andersen's filtration of $F_\lambda(\mathbf{M}_R)$. Fix $0 \le e \le \min(s, r-t)$, and let $a = s-e$, $b = t+e$, $c = s+t$. Then Proposition 4.2.2 implies that $T_R(\lambda(s-e, t+e))$ is a summand of $\mathbf{M}_R(s,t)$ if and only if $\phi_e^\lambda \notin \mathcal{A}^1$; since $\mathcal{J}^1 = \mathcal{A}^1$, this holds if and only if p does not divide $\binom{s+t-2a}{e} = \binom{t-s+2e}{e}$, whence the Proposition. □

6.3. Explicit Howe dual pairs and applications. In this section, we explicitly realize Howe dual data $(G, H, \Gamma, \mathbf{M})$ corresponding to the situations described above in (6.2.e). In each case we define $(G_R, H_R, \Gamma_R, \mathbf{M}_R)$; we then check that the following holds:

6.3.1. $(G, H, \Gamma, \mathbf{M})$ *is a Howe dual datum defined over R. The set of weights of* $res_H^\Gamma(\mathbf{M})$, *the ordered set of dominant weights* $res_G^\Gamma(\mathbf{M})$, *and the order reversing bijection* $'$ *correspond to* $\pi_H, \pi_G^+, '$ *as described in one of the three cases of* (6.2.e).

An explicit description for the identification of π_G^+ with the dominant weights of $\mathrm{res}_G^\Gamma(\mathbf{M})$ will be given. In what follows, S_R denotes a maximal torus of H_R.

6.3.A. *General Linear Case.* Let V_R, W_R be free R-modules of respective ranks r and 2. Put $G_R = \mathrm{GL}(V_R)$, $H_R = \mathrm{GL}(W_R) = \mathrm{GL}_{2,R}$, $\Gamma_R = \mathrm{GL}(V_R \otimes_R W_R)$, and $\mathbf{M}_R = \bigwedge^{\cdot}(V_R \otimes_R W_R)$. According to 5.3.2, G_R, H_R are subgroup schemes of $\mathrm{GL}(W_R)$.

The results in [Don93, §3] yield 6.3.1 for case I of (6.2.e) with the identification $\lambda(a,b) = \varpi_a + \varpi_b$.

Remark 6.3.2. The composition factors for $\Delta_G(\varpi_a + \varpi_b)$ and the Weyl filtration factors for $T_G(\varpi_a + \varpi_b)$ may now be obtained via Proposition 6.2.1.

According to 5.3.2, $\mathrm{res}_{G_R}^{\Gamma_R}(\mathbf{M}) \simeq \bigwedge^{\cdot}(V_R \oplus V_R)$. A straightforward argument shows that $\mathbf{M}_R(s,t)$, corresponds under this isomorphism to $\bigwedge^s V_R \otimes_R \bigwedge^t V_R$ (for a suitable choice of S_R). Thus, Proposition 6.2.5 has the following consequence.

Proposition 6.3.3. *If $0 \le s \le t \le r$, then $\bigwedge^s V \otimes_k \bigwedge^t V \simeq \bigoplus_{e \in Y(s,t)} T(\varpi_{s-e} + \varpi_{t+e})$ as $\mathrm{GL}(V)$ modules, with $Y(s,t) = Y_I(s,t)$ as in Proposition 6.2.5.*

6.3.B. *Symplectic Case.* Let (V_R, α) and (W_R, α') be R-split symplectic spaces of respective ranks $2r$ and 2. Let $G_R = \mathrm{Sp}(V_R)$ and $H_R = \mathrm{Sp}(W_R) \simeq \mathrm{SL}(W_R)$.

As in the remarks preceding 5.3.5, consider the R-split orthogonal space $(V_R \otimes_R W_R, q_R)$; that result shows that G_R and H_R are subgroup schemes of $\Gamma_R = \mathrm{Spin}(q_R)$. Let $\mathbf{M}$ be the spin module for Γ_R.

Applying the results of [AR96], one obtains 6.3.1 for case II of (6.2.e) with the identification $\lambda(a) = \varpi_a$.

Remark 6.3.4. The composition factors for $\Delta_G(\varpi_a)$ and the Weyl filtration factors for $T_G(\varpi_a)$ may now be obtained via Proposition 6.2.2. The composition multiplicities for $\Delta_G(\varpi_a)$ were first obtained by Premet and Suprunenko in [PS83]; in their work, the multiplicities $t(i,j)$ arose by reducing the problem to a question about symmetric group representations for 2-part partitions, and applying the work of James [Jam78].

According to 5.3.5, $\mathrm{res}_{G_R}^{\Gamma_R}\mathbf{M} \simeq \bigwedge^{\cdot} V_R$. It is straightforward to see that under such an isomorphism, $\mathbf{M}(i)$ corresponds to $\bigwedge^i V$; hence Proposition 6.2.6 yields:

Proposition 6.3.5. *If $0 \le s \le r$, then $\bigwedge^{2r-s} V \simeq \bigwedge^s V \simeq \bigoplus_{e \in Y(s)} T(\varpi_e)$ as $\mathrm{Sp}(V)$ modules, with $Y(s) = Y_{II}(s)$ as in Proposition 6.2.6.*

6.3.C. *Spin case.* Let (V_R, q_R) and (W_R, q'_R) denote R-split orthogonal spaces of rank $2r+1$ and 3 respectively, and put $G_R = \mathrm{Spin}(q_R)$, $H_R = \mathrm{Spin}(q'_R) \simeq \mathrm{SL}_{2,R}$. Form the R-split orthogonal space $(V_R \otimes_R W_R, Q_R = q_r \otimes q'_R)$ and let $\Gamma_R = \mathrm{Spin}(Q_R)$. Denote the spin module for G_R by $\mathbb{S}_R^G$, and let $\mathbf{M}_R$ be the spin module for Γ_R. According to 5.3.6, G_R and H_R are subgroup schemes of $\mathrm{Spin}(Q_R)$.

Assume that $p \neq 2$. Applying the results of [AR96], one obtains 6.3.1 for case III of (6.2.e) with the identification $\lambda(a) = \varpi_a + \varpi_r$ for $0 \le a \le r-1$ and $\lambda(r) = 3\varpi_r$.

Remark 6.3.6. For $p > 2$, the composition factors for $\Delta_G(\varpi_a + \varpi_r)$, $\Delta_G(3\varpi_r)$ and the Weyl filtration factors for $T_G(\varpi_a + \varpi_r)$, $T_G(3\varpi_r)$ may now be obtained via Proposition 6.2.3.

According to 5.3.6, $\mathrm{res}_{G_R}^{\Gamma_R}\mathbf{M}_R \simeq \bigwedge^{\cdot} V_R \otimes_R \mathbb{S}_R^{G_R}$. Again for suitable choice of S_R, $\mathbf{M}_R(i)$ corresponds to $\bigwedge^i V_R \otimes_R \mathbb{S}_R^{G_R}$; hence Proposition 6.2.6 gives:

Proposition 6.3.7. *Let $Y(s) = Y_{III}(s)$ be as in Proposition 6.2.6. For $p > 2$, there are isomorphisms of $G = \mathrm{Spin}(q,V)$ modules:*

(a) $\bigwedge^s V \otimes_k \mathbb{S}^G \simeq \bigoplus_{e \in Y(s)} T(\varpi_e + \varpi_r)$ *for* $0 \leq s < r$, *and*

(b) $\bigwedge^r V \otimes_k \mathbb{S}^G \simeq T(3\varpi_r) \oplus \bigoplus_{e \in Y(r),\ e<r} T(\varpi_e + \varpi_r)$.

6.4. Sum formulae. Let $H = SL(W)$ for a two dimensional k vector space W. Suppose that Q is a tilting module for H. It is straightforward to check that the "sum formula" obtained by Andersen in [And97b] takes the form:

$$\sum_{j\geq 1} \dim_{\mathrm{k}} F_a(Q)^j = \sum_{j>0} \nu_p \left(\frac{a+1+j}{j}\right) [Q : \Delta_H(a+2j)]_\Delta. \tag{6.4.a}$$

As in 6.3.B, let $G = \mathrm{Sp}(V)$ where V is a $2r$ dimensional symplectic vector space over k, and let $\mathbf{M}$ be the spin module for $\Gamma = \mathrm{Spin}(V \otimes_k W)$. Thus $(G,H,\Gamma,\mathbf{M})$ forms a Howe dual datum, and $\pi_H, \pi_G^+,'$ are described by case II of (6.2.e). One has

$$[\mathbf{M} : \Delta_H(a+2j)]_\Delta = \dim_k \mathrm{Hom}_G(\Delta_H(a+2j), \mathbf{M}) = \dim_k \Delta_{\mathrm{Sp}(V)}(\varpi_{r-a-2j}).$$

Combining these identities with (6.4.a) and reindexing, one obtains for $\varepsilon = 0,1$:

$$\sum_{j>0} \dim_{\mathrm{k}} \mathcal{J}^j \Delta_{\mathrm{Sp}(V)}(\varpi_{2i+\varepsilon}) = \sum_{k=0}^{i-1} \nu_p \left(\frac{r+1-\varepsilon-i-k}{i-k}\right) \dim_k \Delta_{\mathrm{Sp}(V)}(\varpi_{2k+\varepsilon}) \tag{6.4.b}$$

The formulae (6.4.b) were first obtained by Jantzen in his thesis [Jan73, p. 92–94]; though Jantzen gets an equality of *characters* rather than merely of dimensions (i.e. one may replace $\dim_{\mathrm{k}}(?)$ with $\mathrm{ch}(?)$ on each side of the formula). One may obtain obvious analogous formulae by working in the settings of 6.3.A and 6.3.C.

In view of these formulae, we pose the following questions. Fix $(G,H,\Gamma,\mathbf{M})$ a Howe dual datum defined over R, and let $\lambda \in \pi_G^+$. Consider the space $F = F_\lambda(\mathbf{M})$; according to Theorem 3 we may regard F as the H module $\Delta_H(\lambda')$ and, this identification being made, the Andersen filtration $\mathcal{A}$ of F, and the Jantzen filtration $\mathcal{J}$ of $\Delta_H(\lambda')$, are identical.

Question 1. Consider the sum of H module characters $\sum_{j>0} \mathrm{ch}\mathcal{A}^j F$. Since the Andersen and Jantzen filtrations coincide, the Jantzen sum formulae describes this character sum in terms of the root system of H. Can Andersen's sum formula be adapted to give a formula for this character sum in terms of the root system of G?

Question 2. Andersen's proof of his sum formula stipulates that p exceed the Coxeter number h of the group. Can one utilize the coincidence of the Andersen and Jantzen filtration to eliminate this prime restriction when the tilting module is the module $\mathbf{M}$ associated with the Howe dual datum $(G,H,\Gamma,\mathbf{M})$?

References

[And97a] H. H. Andersen, *Filtrations and tilting modules*, Ann. Sci. École Norm. Sup. **30** (1997), no. 4, 353–366.

[And97b] H. H. Andersen, *A sum formula for tilting filtrations*, Preprint (1997), Preprint 03, Aarhus University.

[And98] H. H. Andersen, *Tilting modules for algebraic groups*, Representation Theory of Algebraic Groups and Related Finite Groups (Cambridge, Jan - Jun 1997), NATO ASI, Isaac Newton Institute for Mathematical Sciences, Kluwer, Dordrecht, 1998.

[AR96] A. M. Adamovich and G. L. Rybnikov, *Tilting modules for classical groups and Howe duality in positive characteristic*, Transform. Groups **1** (1996), no. 1 & 2, 1–33.

[Ben91] David Benson, *Representation theory and cohomology, I. Basic representation theory of finite groups and associative algebras*, Cambridge Stud. Adv. Math., vol. 30, Cambridge Univ. Press, 1991.

[Bor70] Armand Borel, *Properties and linear representations of Chevalley groups*, Seminar on Algebraic Groups and Related Finite Groups (Inst. for Adv. Study, 1968/69) (A. Borel, R. Carter, et al., eds.), Lecture Notes in Math., vol. 131, Springer-Verlag, 1970, pp. A2 – A55.

[Bor91] Armand Borel, *Linear algebraic groups*, 2nd ed., Grad. Texts in Math., no. 129, Springer Verlag, 1991.

[Bou72] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1972.

[CC75] R. Carter and E. Cline, *The submodule structure of Weyl modules for groups of type A_1*, Proceedings of the conference of finite groups, Park City, Utah (Park City, Utah Feb 1975) (William Scott and Fletcher Gross, eds.), Academic Press, 1975, pp. 303–311.

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956.

[CR81] Charles W. Curtis and Irving Reiner, *Methods of representation theory*, vol. 1, Wiley, New York, 1981.

[DG70] M. Demazure and A. Grothendieck, *Schémas en groupes, séminaire de géométrie du Bois Marie 1962/64 (SGA3)*, Lecture Notes in Math., no. 151-153, Springer-Verlag, 1970.

[Don85] S. Donkin, *Rational representations of algebraic groups: Tensor products and filtrations*, Lecture Notes in Math., no. 1140, Springer Verlag, 1985.

[Don86] S. Donkin, *On Schur algebras and related algebras, I*, Journal of Algebra **104** (1986), 310–328.

[Don93] Stephen Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), 39–60.

[DR92] V. Dlab and C. Ringel, *The module-theoretic approach to quasi-hereditary algebras*, Representations of Algebras and Related Topics (H. Tachikawa and S. Brenner, eds.), London Math. Soc. Lecture Note Ser., vol. 168, London Math. Soc., 1992, pp. 200–224.

[Erd94] Karin Erdmann, *Symmetric groups and quasi-hereditary algebras*, Finite dimensional algebras and related topics. Proceedings of the NATO Advanced Research Workshop on Representations of algebras and related topics, (Ottawa, Canada, August, 1992) (V. Dlab et al., eds.), NATO ASI, Math. Phys. Sci., vol. 424, Kluwer, 1994, pp. 123–161.

[How89] Roger Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570.

[Hum80] James Humphreys, *An introduction to Lie algebras and representation theory*, third ed., Grad. Texts in Math., no. 9, Springer Verlag, 1980.

[Jam78] Gordon James, *The representation theory of the symmetric groups*, Lecture Notes in Math., vol. 682, Springer-Verlag, Berlin, 1978.

[Jan73] Jens C. Jantzen, *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen*, Bonner Math. Schr. (1973), no. 67.

[Jan87] Jens C. Jantzen, *Representations of algebraic groups*, Pure and Applied Mathematics, vol. 131, Academic Press, Orlando, FL, 1987.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, Amer. Math. Soc. Colloq. Publ., vol. 44, Amer. Math. Soc., 1998.

[Mat90] Olivier Mathieu, *Filtrations of G-modules*, Ann. Sci. École Norm. Sup. **23** (1990), no. 2, 625–644.

[PS83] Alexander A. Premet and Irena D. Suprunenko, *The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights*, Comm. Algebra (1983), no. 11, 1309–1342.

[Rei75] Irving Reiner, *Maximal orders*, Academic Press, London, 1975.

[Rin91] Claus M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. **208** (1991), 209–223.

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Grad. Texts in Math., vol. 42, Springer Verlag, 1977.

[Ser79] Jean-Pierre Serre, *Local fields*, Grad. Texts in Math., vol. 67, Springer Verlag, 1979.

E-mail address: mcninchg@member.ams.org