ERRATA FOR "ABELIAN UNIPOTENT SUBGROUPS OF REDUCTIVE GROUPS"

GEORGE MCNINCH

Errors in the paper (McNinch 2002):

(i) As in §4.4, consider a parabolic subgroup P of G determined by a cocharacter $\tau : \mathbf{G}_m \to G$. In §4.4, the quantity $n(P)$ is defined as the least $n \geq 0$ with $\mathfrak{g}(2n) = 0$ for the grading of $\mathfrak{g}$ induced by the cocharacter τ. Write $c(\mathfrak{u})$ for the nilpotence class of $\mathfrak{u}$, and $c(U)$ for that of U. Now (McNinch 2002, Prop. 4.4) erroneously asserts that $c(\mathfrak{u})$, $c(U)$ and $n(P)$ coincide; in fact the given proof shows precisely that $n(P) - 1 = c(\mathfrak{u}) = c(U)$. [1]

(ii) The preceding error in (McNinch 2002, Prop. 4.4) led to a flawed statement of the main result of the paper, (McNinch 2002, Theorem 1.1). See Section 1 below for a corrected formulation of Prop. 4.4 and Theorem 1.1

(iii) More generally, throughout the paper, $n(P)$ should always denote the integer of part (a) of Proposition 1. Thus - under the standing hypotheses on the reductive group G - $n(P)$ is given by $c(U) + 1$ or equivalently by the formula at the bottom of p. 278. As just noted, this applies especially to the formulation of Theorem 1.1. Other occurrences of $n(P)$ are:
- statement of Theorem 5.4 (p. 282)
- statement of Theorem 6.2 (p. 284)
- §8, the second sentence on p. 292 and statement of the Corollary.

(iv) The results in §9.7 have the stated hypothesis that "p is a good prime for the group G". Throughout this section, this condition should be replaced by the hypothesis "p is a very good prime for G" [2]. In particular, the Lemma and Proposition found here in §9.7 are not valid for all good primes. See Section 2 below for an example and further discussion.

1. THEOREM 1.1 AND PROPOSITION 4.4

Proposition 1 (Reformulation of Prop. 4.4 of (McNinch 2002))**.** *With P as in §4.4:*

(a) $n(P) - 1 = c(V) = c(\mathfrak{v})$ *where* $\mathfrak{v} = \mathrm{Lie}(V)$.

(b)] If $m \geq 1$ is minimal such that $p^m \geq n(P)$, then a Richardson element in V has order $\leq p^m$ and a Richardson element in $\mathfrak{v}$ has p-nilpotence degree $\leq m$.

Sketch. The proof of Proposition 4.4 given in (McNinch 2002) shows that $C^j(\mathfrak{v}) = \bigoplus_{i \geq 2j} \mathfrak{g}(2i+2)$ for each j. Since $c(\mathfrak{v})$ is the minimal $j \geq 1$ with $C^j(\mathfrak{v}) = 0$, we see that $c(\mathfrak{v}) = n(P) - 1$. The argument for V is the same. Since $n(P)$ exceeds the nilpotence class of V and $\mathfrak{v}$, (b) follows from (a) by applying (McNinch 2002, Lemma 2). □

Theorem 2 (Reformulation of Theorem 1.1 of (McNinch 2002))**.** *Assume that p is a good prime for the connected reductive group G, and that P is a distinguished parabolic subgroup of G with unipotent radical U. Write $c(U)$ for the nilpotence class of U, write $n(P) = c(U) + 1$, and let the integer $m > 0$ be minimal with the property that $p^m \geq n(P)$.*

(a) The p-nilpotence degree of a Richardson element of $\mathrm{Lie}(U)$ *is m; equivalently, the p-exponent of the Lie algebra* $\mathrm{Lie}(U)$ *is m;*

(b) The order of a Richardson element of U is p^m; equivalently, the exponent of U is p^m.

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[1] This error was noted in the footnote found in (McNinch 2003, pf of Lemma 11; p. 44)

[2] In fact, the results of §9.7 hold when G is a *standard reductive group* as in e.g. (McNinch and Testerman 2016, §4).

Sketch. The Theorem is a consequence of (McNinch 2002, Theorems 5.4 and 6.2), which are formulated provided $n(P)$ as defined in (a) of Proposition 1. □

2. RESULTS IN 9.7

In good characteristic, the tangent mapping to the isogeny $\pi : G_{\mathrm{sc}} \to G$ induces a bijection between the respective nilpotent varieties (and even the p-nilpotent varieties) in good characteristic; see (McNinch 2003, §6 and §7). But the proof of Lemma 9.7 requires more than the statement "$d\pi$ induces a bijection"; one needs to know for each abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$ generated by nilpotent elements,

$$(\clubsuit) \quad d\pi^{-1}(\mathfrak{a}) \text{ contains an abelian subalgebra } \mathfrak{a}' \text{ generated by nilpotent elements}$$

This would follow for example if one knew that $d\pi$ induces an isomorphism

$$(\heartsuit) \quad \mathfrak{c}_{\mathrm{Lie}(G_{\mathrm{sc}})}(X) \xrightarrow[d\pi]{\sim} \mathfrak{c}_{\mathrm{Lie}(G)}(d\pi X)$$

for all nilpotent $X \in \mathrm{Lie}(G_{\mathrm{sc}})$.

But as is easily verified, ($\clubsuit$) and ($\heartsuit$) both fail in characteristic 2 when $G_{\mathrm{sc}} = \mathrm{SL}_2$ and $G = \mathrm{PGL}_2$. And it is easy to see that the conclusion of Lemma 9.7 is incorrect for G. Of course, $p = 2$ is "good" but not "very good" for this G. On the other hand, ($\heartsuit$) is valid in very good characteristic, since in that case $d\pi$ is an isomorphism of Lie algebras.

Here is a corrected proof of Lemma 9.7; the given argument (under the assumption that p is very good for G) then confirms Proposition 9.7.

Lemma 3. *Suppose that p is very good for G, and that B is a Borel subgroup of G with unipotent radical U. Let $\mathfrak{a} \subset \mathfrak{g}$ be an Abelian subalgebra generated by nilpotent elements. Then there is $g \in G$ such that $Ad(g)\mathfrak{a} \subset \mathrm{Lie}(U)$.*

Sketch. As note above, when p is very good for $\mathfrak{g}$, the mapping $d\pi$ determines an isomorphism between $\mathrm{Lie}(G_{\mathrm{sc}})$ and $\mathrm{Lie}(G)$ [3]. Now if $\mathfrak{a}$ is an abelian subalgebra of $\mathrm{Lie}(G)$ generated by nilpotent elements, $\mathfrak{a}_{\mathrm{sc}} = d\pi^{-1}(\mathfrak{a})$ is again abelian and generated by nilpotent elements. Moreover, $\mathfrak{a}_{\mathrm{sc}}$ is conjugate to a subalgebra of $\mathrm{Lie}(U_{\mathrm{sc}})$ if and only if $\mathfrak{a}$ is conjugate to a subalgebra of $\mathrm{Lie}(U)$, where U_{sc} is the unipotent radical of the Borel subgroup $B_{\mathrm{sc}} = \pi^{-1}(B)$ of G_{sc}

Thus we may suppose G to be the product of a torus and simply connected quasisimple groups. Now the remainder of the proof proceeds as in the original manuscript. □

REFERENCES

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DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, 503 BOSTON AVENUE, MEDFORD, MA 02155, USA
E-mail address: george.mcninch@tufts.edu, gmcninch@zoho.com

[3] More generally, $d\pi$ is an isomorphism when G is a *standard reductive group* as in (McNinch and Testerman 2016, §4).